EXTENSIVE FORM GAMES

Heinrich H. Nax & Bary S. R. Pradelski

heinrich.nax@uzh.ch bpradelski@ethz.ch

Dynamic games

Many situations (games) are characterized by sequential decisions and information about prior moves

- Market entrant vs. incumbent (think BlackBerry vs. Apple iPhone)
- Chess
- ...

When such a game is written in strategic form, important information about timing and information is lost.

Solution:

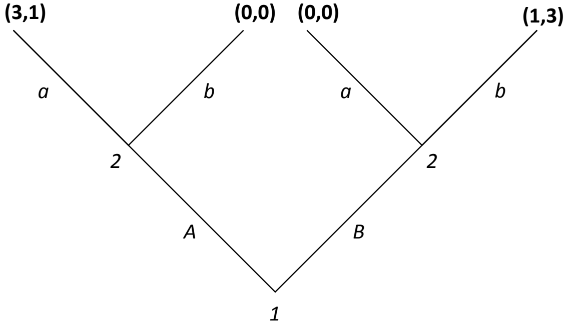
- Extensive form games (via game trees)
- Discussion of timing and information
- New equilibrium concepts

Example: perfect information

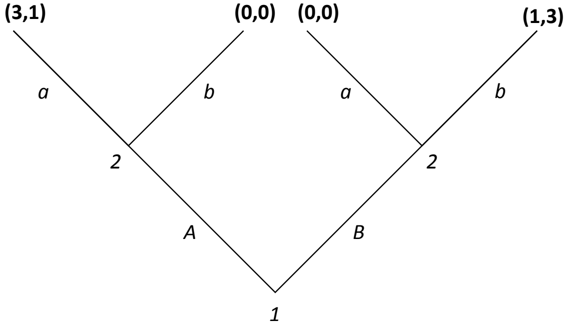
Battle of the sexes:

	a	b
A	3, 1	0, 0
B	0, 0	1, 3

What if row player (player 1) can decide first?

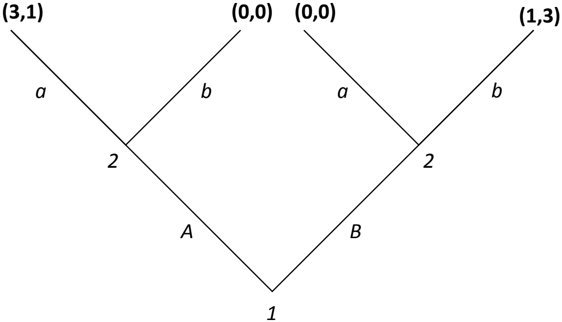


Example: perfect information



What would you do as player 1, A or B?
What would you do as player 2 if player 1 played A, a or b?
What would you do as player 2 if player 1 played B, a or b?

Example: perfect information



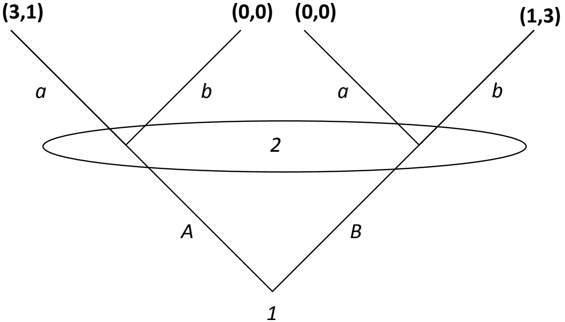
Player 2 would like to commit that if player 1 plays A he will play b (in order to make player 1 play B).

But fighting is not time consistent. Once player 1 played A it is not rational for player 2 to play b.

The expected outcome is A followed by a for payoffs $(3, 1)$.

This is called **backward induction**. It results in a **subgame perfect equilibrium**. More later!

Example: imperfect information



What would you do as player 1, A or B?
What would you do as player 2, a or b?

Timing and information matters!

Extensive form game: Definition

An **extensive-form game** is defined by:

- **Players**, $N = \{1, \ldots, n\}$, with typical player $i \in N$. Note: *Nature* can be one of the players.
- Basic structure is a tree, the **game tree** with nodes $a \in A$. Let a_0 be the root of the tree.
- Nodes are game states which are either
  - **Decision nodes** where some player chooses an action
  - **Chance nodes** where nature plays according to some probability distribution

Representation

Extensive form

- Directed graph with single initial node; edges represent moves
- Probabilities on edges represent Nature moves
- Nodes that the player in question cannot distinguish (information sets) are circled together (or connected by dashed line)

Extensive form → normal form

- A strategy is a player's complete plan of action, listing move at every information set of the player
- Different extensive form games may have same normal form (loss of information on timing and information)

Question: What is the number of a player's strategies?
Product of the number of actions available at each of his information sets.

Subgames (Selten 1965, 1975)

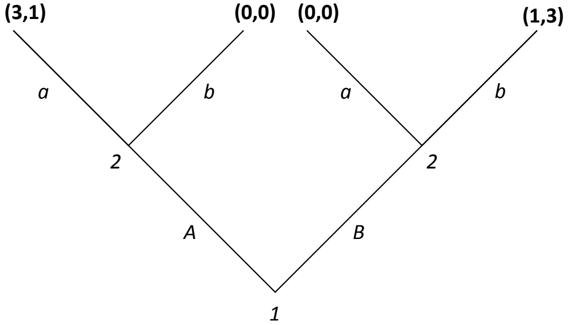
Given a node a in the game tree consider the subtree rooted at a. a is the root of a subgame if

- a is the only node in its information set
- if a node is contained in the subgame then all its successors are contained in the subgame
- every information set in the game either consists entirely of successor nodes to a or contains no successor node to a.

If a node a is a subroot, then each player, when making a choice at any information set in the game, knows whether a has been reached or not. Hence if a has been reached it is as if a “new” game has started.

Subgame examples

How many subgames does the game have?



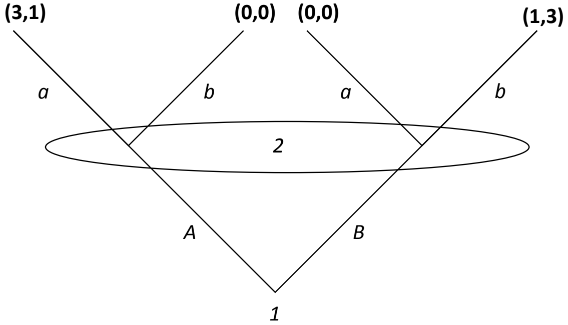
Which strategies does each player have?

Strategies player 1: $\{A, B\}$

Strategies player 2: $\{(a, a), (a, b), (b, a), (b, b)\}$

Subgame examples

How many subgames does the game have?

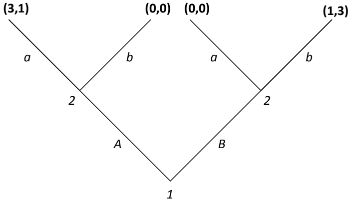


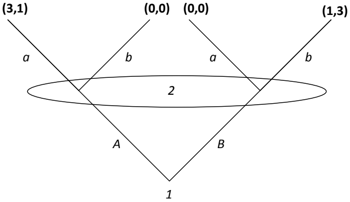
Which strategies does each player have?

Strategies player 1: $\{A, B\}$

Strategies player 2: $\{a, b\}$

Subgame examples: Equivalence to normal form





	a,a	a,b	b,a	b,b
A	3, 1	3, 1	0, 0	0, 0
B	0, 0	1, 3	0, 0	1, 3

	a	b
A	3, 1	0, 0
B	0, 0	1, 3

where columns strategies are of the form *strategy against A, strategy against B*

Strategies in extensive games

PURE STRATEGY s_i One move for each information set of the player.

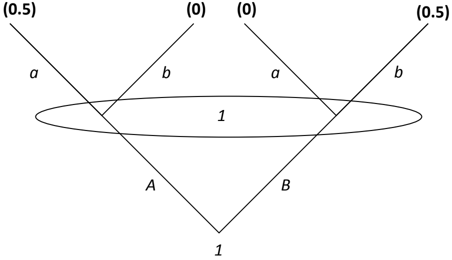
MIXED STRATEGY σ_i Any probability distribution x_i over the set of pure strategies S_i.

BEHAVIOR STRATEGY y_i Select randomly at each information set the move to be made (can delay coin-toss until getting there).

Behavior strategies are special case of a mixed strategy: moves are made with independent probabilities at information sets.

Pure strategies are special case of a behavior strategy.

Example (imperfect recall)



There is one player who has "forgotten" his first move when his second move comes up. (For example: did he lock the door before leaving or not?)

The indicated outcome, with probabilities in brackets, results from the mixed strategy, $\frac{1}{2}Aa + \frac{1}{2}Bb$.

$\Rightarrow$ There exists no behavior strategy that induces this outcome.

The player exhibits "poor memory" / "imperfect recall".

Perfect recall

Perfect recall (Kuhn 1950)

Player i in an extensive form game has *perfect recall* if for every information set h of player i, all nodes in h are preceded by the same sequence of moves of player i.

Kuhn's theorem

Definition: Realization equivalent

A mixed strategy σ_i is *realization equivalent* with a behavior strategy y_i if the realization probabilities under the profile σ_i, σ_{-i} are the same as those under y_i, σ_{-i} for all profiles σ.

Kuhn's theorem

Consider a player i in an extensive form with perfect recall. For every mixed strategy σ_i there exists a realization-equivalent behavior strategy y_i.

Kuhn's Theorem - proof (not part of exam)

Given: mixed strategy σ

Wanted: realization equivalent behavior strategy y

Idea: $y =$ observed behavior under σ

$y(c) =$ observed probability $\sigma(c)$ of making move c.

What is $\sigma(c)$?

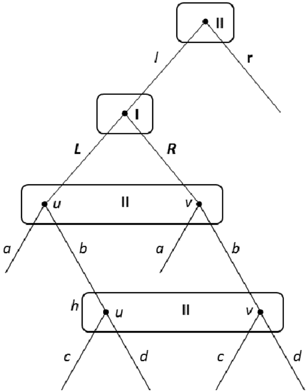
Look at sequence ending in c, here *lbc*.

$\sigma[lbc]$ = probability of *lbc* under $\sigma = \sigma(l, b, c)$.

Sequence *lb* leading to info set h

$$\mu[lb] = \sigma(l, b, c) + \sigma(l, b, d)$$

$$\Rightarrow \sigma[lb] = \sigma[lbc] + \sigma[lbd]$$



Kuhn's Theorem - proof (not part of exam)

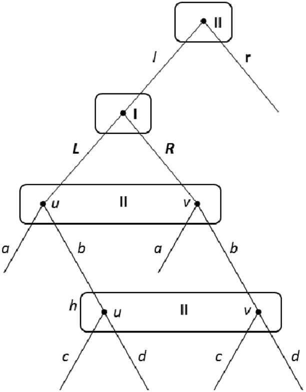
$$\Rightarrow \sigma(c) = \frac{\sigma[lbc]}{\sigma[lb]} =: y(c)$$

$$\Rightarrow \sigma(b) = \frac{\sigma[l]}{\sigma[l]} =: y(b)$$

first info set: $\sigma[\emptyset] = 1 = \sigma[l] + \sigma[r]$

$$\sigma(l) = \frac{\sigma[l]}{\sigma[\emptyset]} =: y(l)$$

$$\Rightarrow y(l)y(b)y(c) = \frac{\sigma[l]}{\sigma[\emptyset]} \cdot \frac{\sigma[l]}{\sigma[l]} \cdot \frac{\sigma[lbc]}{\sigma[lb]} = \sigma[lbc]$$



$\Rightarrow$ $\boxed{y \text{ equivalent to } \sigma}$

Subgame perfect equilibrium

Definition: subgame perfect equilibrium (Selten 1965)

A behavior strategy profile in an extensive form game is a *subgame perfect equilibrium* if for each subgame the restricted strategy is a Nash equilibrium of the subgame.

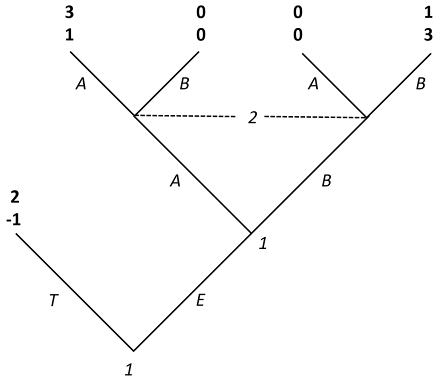
Theorem

Every finite game with prefect recall has at least one subgame perfect equilibrium. Generic such games have a unique subgame perfect equilibrium.

Generic = with probability 1 when payoffs are drawn from continuous independent distributions.

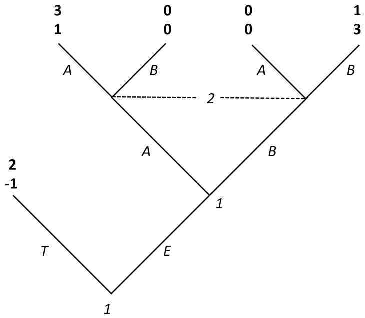
Example: An Outside-option game

Reconsider the battle-of-sexes game (BS game), but player 1 can decide if she joins the game before.



- What are the subgames?
- What are the subgame perfect equilibria?

Example: An Outside-option game



If player 1 decides to enter the BS subgame, player 2 will know that player 1 joint, but will not know her next move.

There exist three subgame perfect equilibria, one for each Nash equilibria of the BS game:

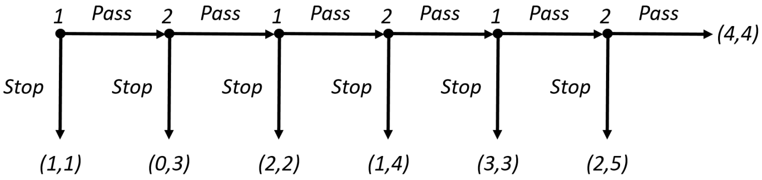
- $S = \{EA, A\}$ Player 1 earns 3, Player 2 earns 1.
- $S = \{TB, B\}$ Player 1 earns 2, Player 2 earns -1.
- $S = \{T(3/4 \cdot A + 1/4 \cdot B), (1/4 \cdot A + 3/4 \cdot B)\}$ Player 1 earns 2, Player 2 earns -1.

Cook-book: Backward induction

"Reasoning backwards in time":

- First consider the last time a decision might be made and choose what to do (that is, find Nash equilibria) at that time
- Using the former information, consider what to do at the second-to-last time a decision might be made
- ...
- This process terminates at the beginning of the game, the found behavior strategies are subgame prefect equilibria

Example: The Centiped game (Rosenthal)



What is the unique subgame perfect equilibirum?

Stop at all nodes.

But in experiments most subjects *Pass* initially: a "trust bubble" forms.

Palacios-Huerta & Volij:

- Chess masters stop right away; students do not…
- …unless they are told they are playing chess masters.

THANKS EVERYBODY

Keep checking the website for new materials as we progress:
http://gametheory.online/project_show/9