NORMAL FORM GAMES: Strategies, dominance, and Nash

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March 12, 2018: Lecture 4
Plan

- Introduction normal form games
- Dominance in pure strategies
- Nash equilibrium in pure strategies
- Best replies
- Dominance, Nash, best replies in mixed strategies
- Nash’s theorem and proof via Brouwer
The Prisoner’s Dilemma

"Two suspects are arrested and interviewed separately. If they both keep quiet (i.e., cooperate) they go to prison for one year. If one suspect supplies evidence (defects) then that one is freed, and the other one is imprisoned for eight years. If both defect then they are imprisoned for five years."

**Players**  The players are the two suspects \( N = \{1, 2\} \).

**Strategies**  The strategy set for player 1 is \( S_1 = \{C, D\} \), and for player 2 it is \( S_2 = \{C, D\} \).

**Payoffs**  For example, \( u_1(C, D) = -8 \) and \( u_2(C, D) = 0 \). All payoffs are represented in this matrix:

\[
\begin{array}{c|cc}
 & \text{Cooperate} & \text{Defect} \\
\hline
\text{Cooperate} & -1, -1 & -8, 0 \\
\text{Defect} & 0, -8 & -5, -5 \\
\end{array}
\]
A normal form (or strategic form) game consists of three objects:

1. **Players**: $N = \{1, \ldots, n\}$, with typical player $i \in N$.
2. **Strategies**: For every player $i$, a finite set of strategies, $S_i$, with typical strategy $s_i \in S_i$.
3. **Payoffs**: A function $u_i : (s_1, \ldots, s_n) \rightarrow \mathbb{R}$ mapping strategy profiles to a payoff for each player $i$. $u : S \rightarrow \mathbb{R}^n$.

Thus a normal form game is represented by the triplet:

$$G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$$
Lecture 4: NORMAL FORM GAMES: Strategies, dominance, and Nash

Strategies

**Definition: strategy profile**

\[ s = (s_1, \ldots, s_n) \]

is called a *strategy profile*. It is a collection of strategies, one for each player. If \( s \) is played, player \( i \) receives \( u_i(s) \).

**Definition: opponents strategies**

Write \( s_{-i} \) for all strategies except for the one of player \( i \). So a strategy profile may be written as \( s = (s_i, s_{-i}) \).
Dominance

A strategy strictly dominates another if it is always better whatever others do.

**Strict Dominance**  A strategy $s_i$ strictly dominates $s'_i$ if $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for all $s_{-i}$.

\begin{table}
\begin{tabular}{c|cc}
   & Cooperate & Defect \\
\hline
Cooperate & $-1, -1$ & $-8, 0$ \\
Defect & $0, -8$ & $-5, -5$ \\
\end{tabular}
\end{table}
Dominance

A strategy strictly dominates another if it is always better whatever others do.

**Strict Dominance**  A strategy $s_i$ strictly dominates $s'_i$ if $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for all $s_{-i}$. 

**Dominated Strategy**  A strategy $s'_i$ is strictly dominated if there is an $s_i$ that strictly dominates it.

**Dominant Strategy**  A strategy $s_i$ is strictly dominant if it strictly dominates all $s'_i \neq s_i$.

If players are rational they should never play a strictly dominated strategy, no matter what others are doing, they may play weakly dominated strategies:

**Weak Dominance**  A strategy $s_i$ weakly dominates $s'_i$ if $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s_{-i}$. 
Dominant-Strategy Equilibrium

**Definition: Dominant-Strategy Equilibrium**

The strategy profile $s^*$ is a *dominant-strategy equilibrium* if, for every player $i$, $u_i(s^*_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategy profiles $s = (s_i, s_{-i})$.

Example: Prisoner’s dilemma

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>$-1, -1$</td>
<td>$-8, 0$</td>
</tr>
<tr>
<td>Defect</td>
<td>$0, -8$</td>
<td>$-5, -5$</td>
</tr>
</tbody>
</table>

$(D, D)$ is the (unique) *dominant-strategy equilibrium*. 
Common knowledge of rationality and the game

Suppose that players are rational decision makers and that mutual rationality is common knowledge, that is:

- I know that she knows that I will play rational
- She knows that “I know that she knows that I will play rational”
- I know that “She knows that “I know that she knows that I will play rational””
- ...

Further suppose that all players know the game and that again is common knowledge.
Iterative deletion of strictly dominated strategies

If the game and rationality of players are common knowledge, iterative deletion of strictly dominated strategies yields the set of “rational” outcomes.

\[
\begin{array}{ccc}
  & L & C & R \\
 T & 1,1 & 2,0 & 2,2 \\
 M & 0,3 & 1,5 & 4,4 \\
 B & 2,4 & 3,6 & 3,0 \\
\end{array}
\]

*Note:* Iteratively deletion of strictly dominated strategies is independent of the order of deletion.
Battle of the Sexes

**Players** The players are the two students \( N = \{ \text{row, column} \} \).

**Strategies** Row chooses from \( S_{\text{row}} = \{ \text{Cafe, Pub} \} \)
Column chooses from \( S_{\text{column}} = \{ \text{Cafe, Pub} \} \).

**Payoffs** For example, \( u_{\text{row}}(\text{Cafe, Cafe}) = 4 \). The following matrix summarises:

\[
\begin{array}{cc}
\text{Cafe} & \text{Pub} \\
\text{Cafe} & 4, 3 & 1, 1 \\
\text{Pub} & 0, 0 & 3, 4 \\
\end{array}
\]
Battle of the Sexes

In this game, nothing is dominated, so profiles like (Cafe, Pub) are not eliminated. Should they be?

- Column player would play Cafe if row player played Cafe!
- Row player would play Pub if column player played Pub!

In other words, after the game, both players may "regret" having played their strategies.

This a truly interactive game – best responses depend on what other players do ... next slides!
Nash Equilibrium

Definition: Nash Equilibrium

A *Nash equilibrium* is a strategy profiles $s^*$ such that for every player $i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } s_i$$

At $s^*$, no $i$ regrets playing $s_i^*$. Given all the other players’ actions, $i$ could not have done better.
Best-reply functions

What should each player do given the choices of their opponents? They should "best reply".

Definition: best-reply function

The best-reply function for player $i$ is a function $B_i$ such that:

$$B_i(s_{-i}) = \{ s_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \}$$
Best-reply functions in Nash

Nash equilibrium can be redefined using best-reply functions:

**Definition: Nash equilibrium**

\[ s^* \text{ is a } \text{Nash equilibrium} \text{ if and only if } s_i^* \in B_i(s_{-i}^*) \text{ for all } i. \]

In words: a Nash equilibrium is a strategy profile of mutual best responses each player picks a best response to the combination of strategies the other players pick.
Example

For the Battle of the Sexes:

- $B_{row}(\text{Cafe}) = \text{Cafe}$
- $B_{row}(\text{Pub}) = \text{Pub}$
- $B_{column}(\text{Cafe}) = \text{Cafe}$
- $B_{column}(\text{Pub}) = \text{Pub}$

So (Cafe, Cafe) is a Nash equilibrium and so is (Pub, Pub) . . .
Cook book: how to find pure-strategy Nash equilibria

The best way to find (pure-strategy) Nash equilibria is to underline the best replies for each player:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>5, 1</td>
<td>2, 0</td>
<td>2, 2</td>
</tr>
<tr>
<td>M</td>
<td>0, 4</td>
<td>1, 5</td>
<td>4, 5</td>
</tr>
<tr>
<td>B</td>
<td>2, 4</td>
<td>3, 6</td>
<td>1, 0</td>
</tr>
</tbody>
</table>
Hawk-dove game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Hawk</th>
<th>Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawk</td>
<td>-2,-2</td>
<td>4,0</td>
</tr>
<tr>
<td>Dove</td>
<td>0,4</td>
<td>2,2</td>
</tr>
</tbody>
</table>

Player 2

<table>
<thead>
<tr>
<th>Hawk</th>
<th>Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2,-2</td>
<td>4,0</td>
</tr>
<tr>
<td>0,4</td>
<td>2,2</td>
</tr>
</tbody>
</table>
Harmony game

<table>
<thead>
<tr>
<th>Company A</th>
<th>Company B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cooperate</td>
</tr>
<tr>
<td>Cooperate</td>
<td>9,9</td>
</tr>
<tr>
<td>Not Cooperate</td>
<td>7,4</td>
</tr>
</tbody>
</table>
A three player game

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>l</td>
<td>r</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td>0, 21, 0</td>
<td>-10, 11, 1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>10, 0, -10</td>
<td>0, 10, 11</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td>1, 11, 10</td>
<td>11, 1, -9</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>-9, 10, 0</td>
<td>1, 20, 1</td>
</tr>
</tbody>
</table>
Matching Pennies

"Each player has a penny. They simultaneously choose whether to put their pennies down heads up (H) or tails up (T). If the pennies match, column receives row’s penny, if they don’t match, row receives columns’ penny."

**Players** The players are $N = \{\text{row}, \text{column}\}$.

**Strategies** Row chooses from $\{H, T\}$; Column from $\{H, T\}$.

**Payoffs** Represented in the strategic-form matrix:

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>T</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

- Best replies are: $B_{\text{row}}(H) = H, B_{\text{row}}(T) = T, B_{\text{column}}(T) = H$, and $B_{\text{column}}(H) = T$
- There is no pure-strategy Nash equilibrium in this game
Randomizing the strategy

Let one player toss her coin and hence play $H$ with probability 0.5 and $L$ with probability 0.5.

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$1, -1$</td>
<td>$-1, 1$</td>
</tr>
<tr>
<td>$T$</td>
<td>$-1, 1$</td>
<td>$1, -1$</td>
</tr>
</tbody>
</table>

Expected utility of column player when playing $H$:

$$\frac{1}{2} \cdot (1) + \frac{1}{2} \cdot (-1) = 0$$

Expected utility of column player when playing $T$:

$$\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot (1) = 0$$

Column is indifferent! He might decide to also toss a coin!
Mixed strategies

**Definition: Mixed strategy**

A *mixed strategy* $\sigma_i$ for a player $i$ is any probability distribution over his or her set $S_i$ of pure strategies. The set of mixed strategies is:

$$\Delta(S_i) = \left\{ x_i \in \mathbb{R}^{\left|S_i\right|}_+ : \sum_{h \in S_i} x_{ih} = 1 \right\}$$
Mixed extension

**Definition: Mixed extension**

The mixed extension of a game $G$ has players, strategies and payoffs: $\Gamma = \langle N, \{S_i\}_{i \in N}, \{U_i\}_{i \in N} \rangle$, where

1. Strategies are probability distributions in the set $\Delta(S_i)$.
2. $U_i$ is player $i$’s expected utility function assigning a real number to every strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$. 

Lecture 4: NORMAL FORM GAMES: Strategies, dominance, and Nash
Mixed Profiles

Suppose player $i$ plays mixed strategy $\sigma_i$ (that is, a list of probabilities). Denote their probability that this places on pure strategy $s_i$ as $\sigma_i(s_i)$. Then:

$$U_i(\sigma) = \sum_s u_i(s) \prod_{j \in N} \sigma_j(s_j)$$

**Definition: opponents’ strategies**

$\sigma_{-i}$ is a vector of mixed strategies, one for each player, except $i$. So $\sigma = (\sigma_i, \sigma_{-i})$. 
Example: Matching pennies

If row player plays \((1, 0)\) what should column play?

If row player plays \((0.3, 0.7)\) what should column play?

If row player plays \((0.5, 0.5)\) what should column play?

*Which mixed strategy should each player use?*
Best-reply function

The definition extends in a straightforward way:

**Definition: best-reply function**

The *best-reply function* for player $i$ is a function $\beta_i$ such that:

$$\beta_i(\sigma_{i-}) = \{ \sigma_i | U_i(\sigma_i, \sigma_{i-}) \geq U_i(\sigma_i', \sigma_{i-}), \text{ for all } \sigma_i' \}$$
Example: Matching pennies

\[
\begin{array}{c|cc}
 & H & T \\
\hline
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

If column player plays \((q, 1 - q)\) what should row play?

- \(U_{row}(H, q) = q - (1 - q) = 2q - 1\), and . . .
- \(U_{row}(T, q) = -q + (1 - q) = 1 - 2q\), so . . .
- play H if \(q > \frac{1}{2}\), play T if \(q < \frac{1}{2}\), and . . .
- indifferent if \(q = \frac{1}{2}\): any \((p, 1 - p)\) will do for the row player!
Best-reply graph
A mixed-strategy Nash equilibrium is a profile $\sigma^*$ such that,

$$U_i(\sigma^*_i, \sigma^*_{-i}) \geq U_i(\sigma_i, \sigma^*_{-i}) \text{ for all } \sigma_i \text{ and } i.$$
Proposition

\[ x \in \Delta(S) \text{ is a Nash equilibrium if } x \in \beta(x). \]

Note that if \( x \in \Delta(S) \) is a mixed Nash equilibrium, then every pure strategy in the support of each strategy \( x_i \) is a best reply to \( x \):

\[ s_i \in \text{supp}(x_i) \Rightarrow s_i \in \beta_i(x) \]
Indifference and Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$1, -1$</td>
<td>$-1, 1$</td>
</tr>
<tr>
<td>$T$</td>
<td>$-1, 1$</td>
<td>$1, -1$</td>
</tr>
</tbody>
</table>

Suppose row player mixes with probability $p$ and $1 - p$ on $H$ and $T$:

$$U_{column}(H, p) = p \cdot (-1) + (1 - p) \cdot (1) = 1 - 2p,$$

$$U_{column}(T, p) = p \cdot (1) + (1 - p) \cdot (-1) = 2p - 1$$

Column player is indifferent when $2p - 1 = 1 - 2p \iff p = \frac{1}{2}$.

Similarly for row player.

The only Nash equilibrium involves both players mixing with probability $\frac{1}{2}$. 
Indifference and Matching Pennies
Battle of the Sexes revisited

**Players** The players are the two students \( N = \{\text{row, column}\} \).

**Strategies** Row chooses from \( S_{\text{row}} = \{\text{Cafe, Pub}\} \)
Column chooses from \( S_{\text{column}} = \{\text{Cafe, Pub}\} \).

**Payoffs** For example, \( u_{\text{row}}(\text{Cafe, Cafe}) = 4 \). The following matrix summarises:

<table>
<thead>
<tr>
<th></th>
<th>Cafe(q)</th>
<th>Pub(1−q)</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe(p)</td>
<td>4,3</td>
<td>1,1</td>
<td>(4q+(1−q))</td>
</tr>
<tr>
<td>Pub(1−p)</td>
<td>0,0</td>
<td>3,4</td>
<td>(3(1−q))</td>
</tr>
<tr>
<td>Expected</td>
<td>3p</td>
<td>(p+4(1−p))</td>
<td></td>
</tr>
</tbody>
</table>

Column chooses \( q = 1 \) whenever \( 3p > p + 4(1−p) \iff 6p > 4 \iff p > \frac{2}{3} \).

Row chooses \( p = 1 \) whenever \( 4q + (1−q) > 3(1−q) \iff 6q > 2 \iff q > \frac{1}{3} \).
There is a mixed Nash equilibrium with $p = \frac{2}{3}$ and $q = \frac{1}{3}$. 
Battle of the Sexes: Expected payoff

<table>
<thead>
<tr>
<th></th>
<th>Cafe(1/3)</th>
<th>Pub(2/3)</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe(2/3)</td>
<td>4,3</td>
<td>1,1</td>
<td>4·1/3+2/3</td>
</tr>
<tr>
<td>Pub(1/3)</td>
<td>0,0</td>
<td>3,4</td>
<td>3·2/3</td>
</tr>
<tr>
<td>Expected</td>
<td>3·2/3</td>
<td>2/3 + 4·1/3</td>
<td></td>
</tr>
</tbody>
</table>

Frequency of play:

<table>
<thead>
<tr>
<th></th>
<th>Cafe(1/3)</th>
<th>Pub(2/3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe(2/3)</td>
<td>2/9</td>
<td>4/9</td>
</tr>
<tr>
<td>Pub(1/3)</td>
<td>1/9</td>
<td>2/9</td>
</tr>
</tbody>
</table>

Expected utility to row player: 2
Expected utility to column player: 2
Cook book: How to find mixed Nash equilibria

- Find all pure strategy NE.

Check whether there is an equilibrium in which row mixes between several of her strategies:

- Identify candidates:
  - If there is such an equilibrium then each of these strategies must yield the same expected payoff given column’s equilibrium strategy.
  - Write down these payoffs and solve for column’s equilibrium mix.
  - Reverse: Look at the strategies that column is mixing on and solve for row’s equilibrium mix.

- Check candidates:
  - The equilibrium mix we found must indeed involve the strategies for row we started with.
  - All probabilities we found must indeed be probabilities (between 0 and 1).
  - Neither player has a positive deviation.
Nash’s equilibrium existence theorem

**Theorem (Nash 1951)**

Every finite game has at least one [Nash] equilibrium in mixed strategies.

Original paper is this week’s reading.
Nash’s contribution – remarks

- Put economics beyond one branch of social sciences but allowed it to today encompass all analytical fields in social sciences and beyond
  - political sciences: strategic interactions, contracts, ...
  - biology: evolution
  - economics: auctions, trading, contracts, ...
  - computer sciences: cloud computing, car routing, ...
  - sociology: opinion formation, political polarization, ...
  - ...

- Nash recognized that his equilibrium concept can be used to study
  - non-cooperative games
  - cooperative games – bargaining
  - does not need to assume perfect rationality – mass-action interpretation and evolutionary game theory
THANKS EVERYBODY
See you next week!
And keep checking the website for new materials as we progress:
http://www.coss.ethz.ch/education/GT.html
Voluntary extra reading: Proof of Nash’s theorem; not part of exam
Brouwer’s fixed point theorem

Theorem (Brouwer)

Given \( S \subset \mathbb{R}^n \) convex and compact (bounded and closed), \( f : S \rightarrow S \) continuous. Then \( f \) has at least one fixed point \( s \in S \) with \( f(s) = s \).

Example \( S = [0, 1] \)
Puzzle: the football which cannot be moved

**Theorem (Brouwer)**

Given $S \subset \mathbb{R}^n$ convex and compact (bounded and closed), $f : S \to S$ continuous. Then $f$ has at least one fixed point $s \in S$ with $f(s) = s$.

Can you move a football on its spot such that no point on its sphere (surface) remains in the same spot?
Proof of Nash via Bouwer

The polyhedron \( \Delta(S) \) is non-empty, convex and compact.

Hence, by Bouwer, every continuous function that maps \( \Delta(S) \) into itself has at least one fix point.

We thus have to find a continuous function \( f : \Delta(S) \rightarrow \Delta(S) \) such that every fix point under \( f \) is a Nash equilibrium.
Nash’s construction

For each player $i$ and strategy profile $\sigma$ define the excess payoff player $i$ receives when playing pure strategy $h \in S_i$ in comparison with $\sigma_i$

$$v_{ih}(\sigma) = \max\{0, U_i(e^h_i, \sigma_{-i}) - U_i(\sigma)\}$$

where $e^h_i$ is the unit vector with position $h$ equal to 1.

Let for all $i \in N, h \in S_i$:

$$f_{ih}(\sigma) = \frac{1 + v_{ih}(\sigma)}{1 + \sum_{k \in S_i} x_{ik}v_{ik}(\sigma)}x_{ih}$$

where $\sigma_i = (x_{i1}, x_{i2}, \ldots, x_{i|S_i|})$. 
Nash’s construction

\[ f_{ih}(\sigma) = \frac{1 + v_{ih}(\sigma)}{1 + \sum_{k \in S_i} x_{ik} v_{ik}(\sigma)} x_{ih} \]

We have

- \( f_{ih}(\sigma) \geq 0 \)
- \( \sum_h f_{ih}(\sigma) = 1 \) for all \( i \in N \) and \( \sigma \in \Delta(S) \)
- \( f_{ih}(\sigma) \) is continuous in \( \sigma \)

Thus \( f \) is a continuous mapping of \( \Delta(S) \) to itself

\[ \Rightarrow f \text{ has at least one fix point} \]
Nash’s construction

Suppose that \( \sigma \) is a fixpoint of \( f \), that is \( \sigma = f(\sigma) \). We must have

\[
0 = f_{ih}(\sigma) - x_{ih}
\]

\[
= \frac{1 + v_{ih}(\sigma)}{1 + \sum_{k \in S_i} x_{ik} v_{ik}(\sigma)} x_{ih} - x_{ih}
\]

\[
= \frac{x_{ih} + v_{ih}(\sigma) x_{ih} - x_{ih} - x_{ih} \sum_{k \in S_i} x_{ik} v_{ik}(\sigma)}{1 + \sum_{k \in S_i} x_{ik} v_{ik}(\sigma)}
\]

\[
= [v_{ih}(\sigma) - \sum_{k \in S_i} x_{ik} v_{ik}(\sigma)] x_{ih} = 0
\]

for all \( i \in N, h \in S_i \).
Nash’s construction: fixpoint $\iff$ equilibrium

\[ [v_{ih}(\sigma) - \sum_{k \in S_i} x_{ik}v_{ik}(\sigma)]x_{ih} = 0 \]

“$\Rightarrow$”: This equation is satisfied for $v_{ih}(\sigma) = 0$ for all $i \in N, h \in S_i$, that is, $\sigma$ is a [Nash] equilibrium.

“$\Leftarrow$”: Suppose the equation is satisfied by some $\sigma \in \Delta(S)$ which is not a Nash equilibrium:

\[ v_{ih}(\sigma) = \sum_{k \in S_i} x_{ik}v_{ik}(\sigma) \]

for all $i, h$ with $x_{ih} > 0$.

But this implies that $v_{ih} = 0$ for all such $i, h$, since otherwise all used pure strategies would earn above average, an impossibility.