# COOPERATIVE GAME THEORY: Matching Intermezzo


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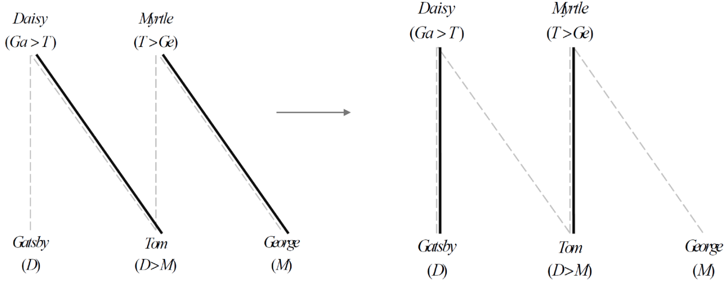
# Stable Marriage/Matching problem



# Stable Marriage/Matching problem

**2-sided market**

- *Men* $M = \{m_1, ..., m_n\}$ on one side, *women* $W = \{w_1, ..., w_n\}$ on the other.
- Each $m_i$: preferences (e.g. $w_1 \succ w_2 \succ ... \succ w_n$) over women
- Each $w_i$: preferences (e.g. $m_n \succ m_1 \succ ... \succ m_{n-1}$) over men

We want to establish a stable matching: forming couples (man-woman) such that there exists no alternative couple where both partners prefer to be matched with each other rather than with their current partners.

# Deferred acceptance

**Gale-Shapley 1962**

*For any marriage problem, one can make all matchings stable using the deferred acceptance algorithm.*

Widely used in practice (e.g. Roth & Sotomayor 1990, Roth et al. ...):

- Resource allocations/doctor recruitment for hospitals
- Organ transplantations
- School admissions/room allocation
- Assigning users to servers in distributed Internet services
- ...

# DA "pseudo-code"

**Initialize** : all $m_i \in M$ and all $w_i \in W$ are *single*.

**Engage** : Each single man $m \in M$ *proposes* to his *preferred woman* $w$ to whom he has *not yet proposed*.

- If $w$ is single, she will become *engaged* with her *preferred proposer*.
- Else $w$ is already engaged with $m'$.
  - If $w$ prefers her preferred proposer $m$ over her current engagement $m'$, then $(m, w)$ become engaged and $m'$ becomes single.
  - Else $(m', w)$ remain engaged.
- All proposers who do not become engaged remain single.

**Repeat** : If there exists a single man after **Engage**, repeat **Engage**; Else move to **Terminate**.

**Terminate** : *Marry* all engagements.

# Proof sketch

**Trade up** : Women can *trade up* until every woman (hence also every man) is engaged, which is when they all get married.

**Termination** : No singles can remain, because every man would eventually propose to every woman as long as he remains single, and every single woman, once proposed to, becomes engaged.

**Termination with stability?**

# Proof sketch

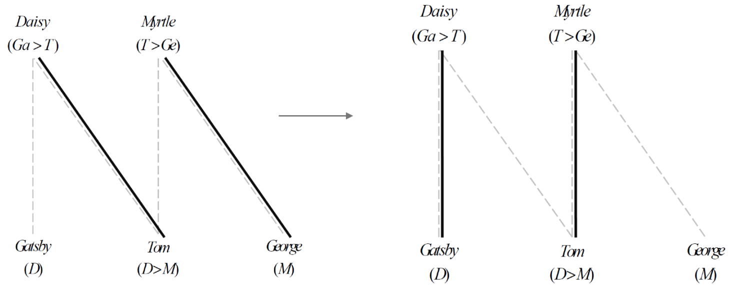
**Stability** : *The resulting matching is stable.*

*Proof* : Suppose the algorithm terminates so that there exists a pair $(m, w)$ whose partners are engaged to $w' \neq w$ and $m' \neq m$ respectively.

Claim : It is not possible for both $m$ and $w$ to prefer each other over their engaged partner. because

- If $m$ prefers $w$ over $w'$, then he proposed to $w$ before he proposed to $w'$. At that time,
  - *Case 1:* If $w$ got engaged with $m$, but did not marry him, then $w$ must have traded up and left $m$ for someone she prefers over $m$, and therefore cannot prefer $m$ over $m'$.
  - *Case 2:* Else, if $w$ did not get engaged with $m$, then she was already with someone she prefers to $m$ at that time, and can therefore not prefer $m$ over $m'$.
- Hence, either $m$ prefers $w'$ over $w$, or $w$ prefers $m'$ over $m$.

# Back to the Great Gatsby



THANKS EVERYBODY

Keep checking the website for new materials as we progress:

http://gametheory.online/project_show/9

Original Gale-Shapley paper:

https://www.eecs.harvard.edu/cs286r/courses/fall09/papers/galeshapley.pdf

GS Algorithm implementation:

https://towardsdatascience.com/gale-shapley-algorithm-simply-explained-caa344e643c2