COOPERATIVE GAME THEORY:
Matching Intermezzo

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Stable Marriage/Matching problem
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2-sided market

- Men $M = \{m_1, \ldots, m_n\}$ on one side, women $W = \{w_1, \ldots, w_n\}$ on the other.
- Each $m_i$: preferences (e.g. $w_1 \succ w_2 \succ \ldots \succ w_n$) over women
- Each $w_i$: preferences (e.g. $m_n \succ m_1 \succ \ldots \succ m_{n-1}$) over men

We want to establish a stable matching: forming couples (man-woman) such that there exists no alternative couple where both partners prefer to be matched with each other rather than with their current partners.
Deferred acceptance

Gale-Shapley 1962

*For any marriage problem, one can make all matchings stable using the deferred acceptance algorithm.*

Widely used in practice (e.g. Roth & Sotomayor 1990, Roth et al. ...):

- Resource allocations/doctor recruitment for hospitals
- Organ transplantations
- School admissions/room allocation
- Assigning users to servers in distributed Internet services
- ...
DA “pseudo-code”

**Initialize**: all $m_i \in M$ and all $w_i \in W$ are single.

**Engage**: Each single man $m \in M$ proposes to his preferred woman $w$ to whom he has not yet proposed.
- If $w$ is single, she will become engaged with her preferred proposer.
- Else $w$ is already engaged with $m'$.
  - If $w$ prefers her preferred proposer $m$ over her current engagement $m'$, then $(m, w)$ become engaged and $m'$ becomes single.
  - Else $(m', w)$ remain engaged.
- All proposers who do not become engaged remain single.

**Repeat**: If there exists a single man after Engage, repeat Engage; Else move to **Terminate**.

**Terminate**: Marry all engagements.
Proof sketch

Trade up: Women can *trade up* until every woman (hence also every man) is engaged, which is when they all get married.

Termination: No singles can remain, because every man would eventually propose to every woman as long as he remains single, and every single woman, once proposed to, becomes engaged.

Termination with stability?
Proof sketch

**Stability** : *The resulting matching is stable.*

**Proof** : Suppose the algorithm terminates so that there exists a pair $(m, w)$ whose partners are engaged to $w' \neq w$ and $m' \neq m$ respectively.

**Claim** : It is not possible for both $m$ and $w$ to prefer each other over their engaged partner. because

- If $m$ prefers $w$ over $w'$, then he proposed to $w$ before he proposed to $w'$. At that time,
  - *Case 1:* If $w$ got engaged with $m$, but did not marry him, then $w$ must have traded up and left $m$ for someone she prefers over $m$, and therefore cannot prefer $m$ over $m'$.
  - *Case 2:* Else, if $w$ did not get engaged with $m$, then she was already with someone she prefers to $m$ at that time, and can therefore not prefer $m$ over $m'$.

- Hence, either $m$ prefers $w'$ over $w$, or $w$ prefers $m'$ over $m$. 

Back to the Great Gatsby
THANKS EVERYBODY
Keep checking the website for new materials as we progress:
http://gametheory.online/project_show/9

Original Gale-Shapley paper:
https://www.eecs.harvard.edu/cs286r/courses/fall09/papers/galeshapley.pdf

GS Algorithm implementation: