COOPERATIVE GAME THEORY: Core and Shapley Value

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The **two** branches of game theory

**Non-cooperative** game theory
- No binding contracts can be written
- Players are individuals
- Nash equilibrium

**Cooperative** game theory
- Binding contract can be written
- Players are individuals and coalitions of individuals
- Main solution concepts:
  - Core
  - Shapley value
- **The focus of today!**
Reminder: the ingredients of a noncooperative game

- **Players**: $N = \{1, 2, \ldots, n\}$
- **Actions / strategies**: each player chooses $s_i$ from his own finite strategy set; $S_i$ for each $i \in N$
  - **Outcome**: resulting strategy combination: $s = (s_1, \ldots, s_n) \in (S_i)_{i \in N}$
- **Payoff outcome**: payoffs $u_i = u_i(s)$
The Theory of Games and Economic Behavior (1944)

John von Neumann (1903-1957) and Oskar Morgenstern (1902-1977)
The ingredients of a cooperative game

- **Population of players**: \(N = \{1, 2, \ldots, n\}\) (finite)
- **Coalitions**: \(C \subseteq N\) form in the population and become players resulting in a coalition structure \(\rho = \{C_1, C_2, \ldots, C_k\}\)
- **Payoffs**: *we still need to specify how* payoffs \(\phi = \{\phi_1, \ldots, \phi_n\}\) come about:
  - something like this: \(\phi_i = \phi(\rho, \text{"sharing rule"})\)
Cooperative games in **characteristic function form (CFG)**

- **The game:** A CFG defined by 2-tuple \( G(v, N) \)
- **Players:** \( N = 1, 2, ..., n \) (finite, fixed population)
- **Coalitions:** disjoint \( C \subseteq N \) form resulting in a coalition structure/partition \( \rho \)
  - \( \emptyset \) is an *empty coalition*
  - \( N \) is the *grand coalition*
  - The set of all coalitions is \( 2^N \)
  - \( \rho \) is the set of all partitions
- **Characteristic function:** \( v \) is the characteristic function form that assigns a *worth* \( v(C) \) to each coalition
  - \( v: 2^N \rightarrow R \)
  - (and \( v(\emptyset) = 0 \))
3-player example

- $N=1,2,3$
- $v(i)=0$
- $v(1,2)=v(1,3)=0.5$
- $v(2,3)=0$
- $v(N)=1$
“Transferable utility” and feasibility

- **The game**: CFG defined by 2-tuple $G(v, N)$
- **Outcome**: *Coalition structure*
  - *partition* $\rho = \{C_1, C_2, \ldots, C_k\}$ and
  - *payoff allocation/imputation* $\phi = \{\phi_1, \ldots, \phi_n\}$
- Importantly, $v(C)$ can be “shared” amongst $i \in C$ (transfer of utils)!
- **Feasibility**: in each $C$, $\sum_{i \in C} \phi_i \leq v(C)$
3-player example: some feasible outcomes

- **Outcome 1**: \( \{(1,2),3\} \) and \( \{(0.25,0.25),0\} \)
- **Outcome 2**: \( \{N\} \) and \( \{0.25,0.25,0.5\} \)
- **Outcome 3**: \( \{N\} \) and \( \{0.8,0.1,0.1\} \)
Superadditivity assumption

Superadditivity

If two coalitions $C$ and $S$ are disjoint (i.e. $S \cap C = \emptyset$), then $v(C) + v(S) \leq v(C \cup S)$

i.e. “mergers of coalitions weakly improve their worths”

- Superadditivity implies efficiency of the grand coalition: for all $\rho \in \rho$, $v(N) \geq \sum_{C \in \rho} v(C)$.

- In our example:
  
  $v(N) > v(1, 2) = v(1, 3) > v(2, 3) = v(1) = v(2) = v(3)$. 

\[
\begin{bmatrix}
1 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The Core (Gillies 1959)

The Core of a superadditive \( G(v, n) \) consists of all outcomes where the grand coalition forms and payoff allocations \( \phi^* \) are

Pareto-efficient: \( \sum_{i \in N} \phi^*_i = v(N) \)

Unblockable: for all \( C \subset N \), \( \sum_{i \in C} \phi^*_i \geq v(C) \)

- individual rational: \( \phi^*_i \geq v(i) \) for all \( i \)
- coalitional rational: \( \sum_{i \in C} \phi^*_i \geq v(C) \) for all \( C \)
3-player example

- **Outcome 1**: \{((1,2),3) \text{ and } (0.25,0.25),0\}
- **Outcome 2**: \{N\} \text{ and } \{0.25,0.25,0.5\}
- **Outcome 3**: \{N\} \text{ and } \{0.8,0.1,0.1\}
Properties of the Core

- A system of weak **linear** inequalities defines the Core, which is therefore **closed** and **convex**.
- The core can be
  - empty
  - non-empty
  - large
Core empty

- $v(i) = 0$
- $v(i,j) = 0.9$
- $v(N) = 1$
Core unique

- $v(i) = 0$
- $v(i,j) = 2/3$
- $v(N) = 1$
Core large

\begin{itemize}
  \item \( v(i) = v(i,j) = 0 \)
  \item \( v(N) = 1 \)
\end{itemize}
Bondareva-Shapley Theorem

**Bondareva 1963 and Shapley 1967**

The Core of a cooperative game is *nonempty* if and only if the game is *balanced*.

**Balancedness:**

- **Balancing weight:** Let $\alpha(C) \in [0,1]$ be the balancing weight attached to any $C \in 2^N$.
- **Balanced family:** A set of balancing weights $\alpha$ is a balanced family if, for every $i$, $\sum_{C \in 2^N : i \in C} \alpha(C) = 1$
- Balancedness in a superadditive game then requires that, for all balanced families, $v(N) \geq \sum_{C \in 2^N} \alpha(C)v(C) > 1$
Limitations of the Core
1. Core empty

- $v(i) = 0$
- $v(i, j) = \frac{5}{6}$
- $v(N) = 1$
2. Core non-empty but very inequitable $(1, 0, 0)$

- $v(i) = v(2, 3) = 0$
- $v(N) = v(1, 2) = v(1, 3) = 1$
3. Core large (any split of 1)

- \( v(i) = v(i, j) = 0 \)
- \( v(N) = 1 \)
So is the Core a *descriptive* or a *prescriptive/normative* solution concept?
What about an explicitly *normative* solution concept?
Lloyd Shapley (1923-2016)
Shapley value (Shapley 1953)

Axioms. Given some $G(v, N)$, an acceptable allocation/value $x^*(v)$ should satisfy

- **Efficiency.** $\sum_{i \in N} x^*_i(v) = v(N)$
- **Symmetry.** If, for any two players $i$ and $j$, $v(S \cup i) = v(S \cup j)$ for all $S$ not including $i$ and $j$, then $x^*_i(v) = x^*_j(v)$
- **Dummy player.** If, for any $i$, $v(S \cup i) = v(S)$ for all $S$ not including $i$, then $x^*_i(v) = 0$
- **Additivity.** If $u$ and $v$ are two characteristic functions, then $x^*(v + u) = x^*(v) + x^*(u)$

$\sum_{i \in N} x^*_i(v) = v(N)$
Cooperative Game Theory

Shapley’s characterization

The unique function satisfying all four axioms for the set of all games is

\[
\phi_i(v) = \sum_{S \in \mathcal{N}, i \in S} \frac{(|S| - 1)! (n - |S|) !}{n!} [v(S) - v(S \setminus \{i\})]
\]

So what does this function mean?
Shapley value

The Shapley value pays each player his *average marginal contributions*:

- For any $S$: $i \in S$, think of the *marginal contribution* $MC_i(S) = v(S) - v(S \setminus i)$.
- And of $\sum_{S \in N, i \in S} \frac{(|S| - 1)! (n - |S|)!}{n!}$ as some kind of “average” operator (more detail later).

Then,

\[
\phi_i(v) = \text{average} \left( MC_i(S) \right)
\]
An alternative characterization

Young (1985): a set of equivalent axioms is

- **Efficiency.** $\sum_{i \in N} x_i^*(v) = v(N)$
- **Symmetry.** if, for any two players $i$ and $j$, $v(S \cup i) = v(S \cup j)$ for all $S$ not including $i$ and $j$, then $x_i^*(v) = x_j^*(v)$
- **Monotonicity.** If $u$ and $v$ are two characteristic functions and, for all $S$ including $i$, $u(S) \geq v(S)$, then $x_i^*(u) \geq x^*(v)$

A more attractive set of axioms...
1. Core empty

- $v(i) = 0$
- $v(i,j) = \frac{5}{6}$
- $v(N) = 1$

Shapley value

$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
2. Core non-empty but very inequitable \((1, 0, 0)\)

- \(v(i) = v(2, 3) = 0\)
- \(v(N) = v(1, 2) = v(1, 3) = 1\)

<table>
<thead>
<tr>
<th>Shapley value</th>
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<tbody>
<tr>
<td>((4/6, 1/6, 1/6))</td>
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</table>
3. Core large (any split of 1)

- $v(i) = v(i,j) = 0$
- $v(N) = 1$

Shapley value

$(1/3, 1/3, 1/3)$
Room-entering story (Roth 1983)

Average MC in this sense...

- \( N = \{1, 2, \ldots, n\} \) players enter a room in some order.
- Whenever a player enters a room, and players \( S \setminus i \) are already there, he is paid his marginal contribution \( MC_i(S) = v(S) - v(S \setminus i) \).
- Suppose all \( n! \) orders are equally likely.
- Then there are \( (s - 1)! \) different orders in which these players in \( S \setminus i \) can precede \( i \)
- and \( (n - s)! \) order in which the others may follow
- hence, a total of \( (s - 1)!(n - s)! \) orders for that case of the \( n! \) total orders.
- Ex ante, the payoff of a players is \( \sum_{S \in N, i \in S} \frac{(s-1)!(n-s)!}{n!} MC_i(S) \) – the Shapley value.
Relationship between the Core and the Shapley value

Put simply, none...

- the Shapley value is normative
- the Core is something else (hybrid)
- when the Core is non-empty, the SV may lie inside or not
- when the Core is empty, the SV is still uniquely determined
Other cooperative models
Non-transferable-utility cooperative game

- As before: CFG defined by 2-tuple $G(v, N)$
- **Outcome**: partition $\rho = \{C_1, C_2, ..., C_k\}$ directly (w/o negotiating how to share) implies a payoff allocation/imputation $\phi_i = f_i(C_i)$
- There are no side-payments and the worth of a coalition cannot be (re-)distributed.

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Agents have preferences over coalitions.
Stable Marriage/Matching problem

2-sided market

- Men $M = \{m_1, \ldots, m_n\}$ on one side, women $W = \{w_1, \ldots, w_n\}$ on the other.
- Each $m_i$: preferences (e.g. $w_1 \succ w_2 \succ \ldots \succ w_n$) over women
- Each $w_i$: preferences (e.g. $m_n \succ m_1 \succ \ldots \succ m_{n-1}$) over men
THANKS EVERYBODY
and keep checking the website for new materials as we progress!
http://gametheory.online