COOPERATIVE GAME THEORY:
Core and Shapley Value

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The **two** branches of game theory

Non-cooperative game theory

- No binding contracts can be written
- Players are individuals
- Nash equilibrium

Cooperative game theory

- Binding contract can be written
- Players are individuals and coalitions of individuals
- Main solution concepts:
  - Core
  - Shapley value
- **The focus of today!**

Reminder: the ingredients of a **noncooperative game**

- **Players**: $N = \{1, 2, ..., n\}$
- **Actions / strategies**: each player chooses s_i from his own finite strategy set; S_i for each $i \in N$
  - **Outcome**: resulting strategy combination: $s = (s_1, ..., s_n) \in (S_i)_{i \in N}$
- **Payoff outcome**: payoffs $u_i = u_i(s)$

The Theory of Games and Economic Behavior (1944)

John von Neumann (1903-1957) and Oskar Morgenstern (1902-1977)

The ingredients of a **cooperative game**

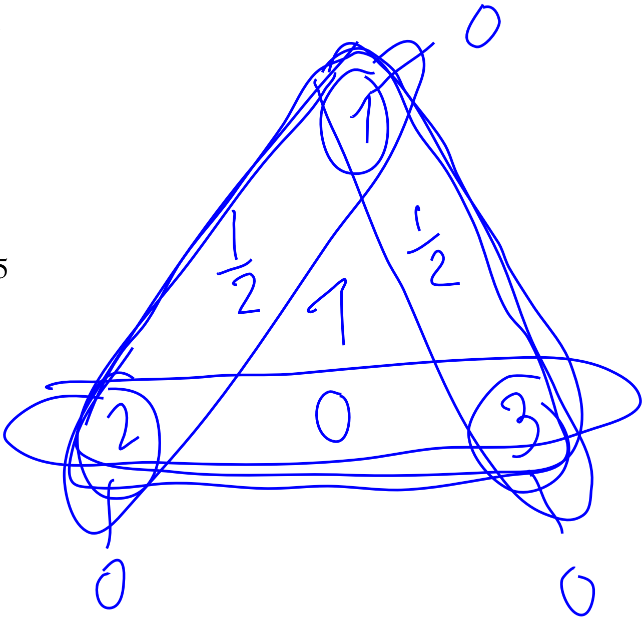
- **Population of players**: $N = \{1, 2, ..., n\}$ (finite)
- **Coalitions**: $C \subseteq N$ form in the population and become players
  - resulting in a coalition structure $\rho = \{C_1, C_2, ..., C_k\}$
- **Payoffs**: *–we still need to specify how–* payoffs $\phi = \{\phi_1, ..., \phi_n\}$ come about:
  - something like this: $\phi_i = \phi(\rho,$"**sharing rule**"$)$

Cooperative games in **characteristic function form** (CFG)

- **The game:** A CFG defined by 2-tuple $G(v, N)$
- **Players:** $N = 1, 2, ..., n$ (finite, fixed population)
- **Coalitions**: *disjoint* $C \subseteq N$ form resulting in a coalition structure/ partition ρ
  - $\emptyset$ is an *empty coalition*
  - N is the *grand coalition*
  - The set of all coalitions is 2^N
  - $\boldsymbol{\rho}$ is the set of all partitions
- **Characteristic function**: v is the characteristic function form that assigns a *worth* $v(C)$ to each coalition
  - v: $2^N \to R$
  - (and $v(\emptyset) = 0$)

3-player example

- N=1,2,3
- v(i)=0
- v(1,2)=v(1,3)=0.5
- v(2,3)=0
- v(N)=1

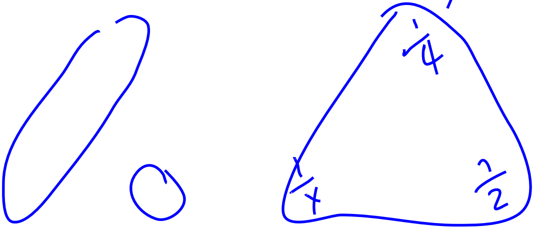


“Transferable utility” and **feasibility**

- **The game:** CFG defined by 2-tuple $G(v, N)$
- **Outcome:** *Coalition structure*
  - *partition* $\rho = \{C_1, C_2, ..., C_k\}$ and
  - *payoff allocation*/imputation $\phi = \{\phi_1, ..., \phi_n\}$
- Importantly, v(C) can be “shared” amongst $i \in C$ (**transfer of utils**)!
- **Feasibility**: in each C, $\sum_{i \in C} \phi_i \leq v(C)$

3-player example: some feasible outcomes

- Outcome 1: {(1,2),3} and {(0.25,0.25),0}
- Outcome 2: {N} and {0.25,0.25,0.5}
- Outcome 3: {N} and {0.8,0.1,0.1}



Superadditivity assumption

Superadditivity

If two coalitions C and S are disjoint (i.e. $S \cap C = \emptyset$), then $v(C) + v(S) \leq v(C \cup S)$

i.e. "mergers of coalitions weakly improve their worths"

- Superadditivity implies *efficiency* of the grand coalition: for all $\rho \in \boldsymbol{\rho}$, $v(N) \geq \sum_{C \in \rho} v(C)$.
- In our example:
  $v(N) > v(1,2) = v(1,3) > v(2,3) = v(1) = v(2) = v(3)$.
  1 0.5 0.5 0 0 0 0

The Core (Gillies 1959)

The Core

The Core of a superadditive $G(v, n)$ consists of all outcomes where the *grand coalition* forms and payoff allocations ϕ^* are

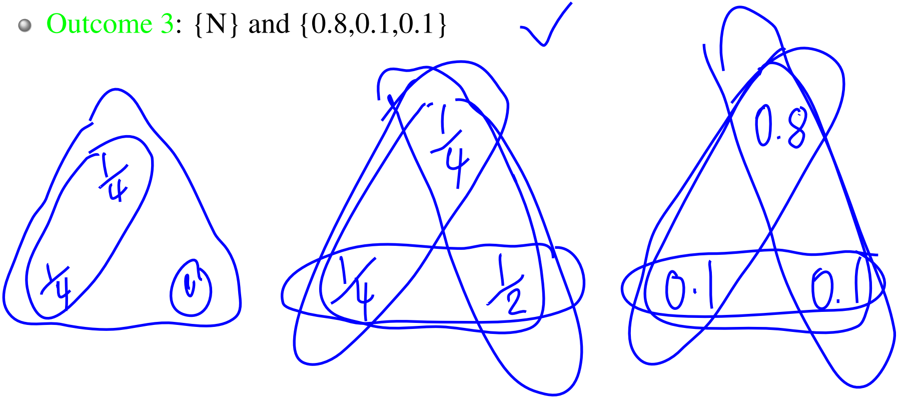
Pareto-efficient: $\sum_{i \in N} \phi_i^* = v(N)$

Unblockable: for all $C \subset N$, $\sum_{i \in C} \phi_i^* \geq v(C)$

- *individual rational*: $\phi_i^* \geq v(i)$ for all i
- *coalitional rational*: $\sum_{i \in C} \phi_i^* \geq v(C)$ for all C

3-player example

- Outcome 1: {(1,2),3} and {(0.25,0.25),0}
- Outcome 2: {N} and {0.25,0.25,0.5}
- Outcome 3: {N} and {0.8,0.1,0.1}

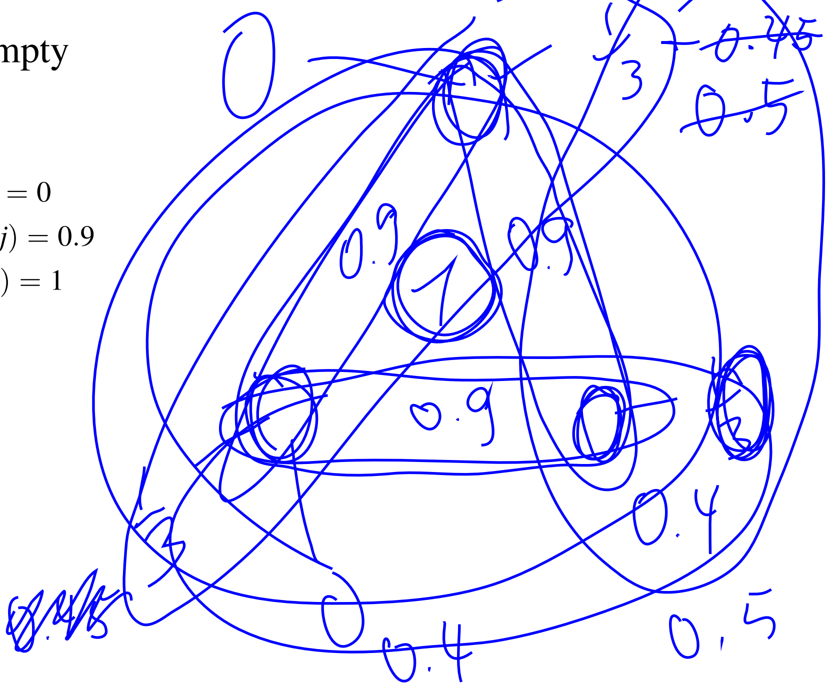


Properties of the Core

- A system of weak linear inequalities defines the Core, which is therefore closed and convex.
- The core can be
  - empty
  - non-empty
  - large

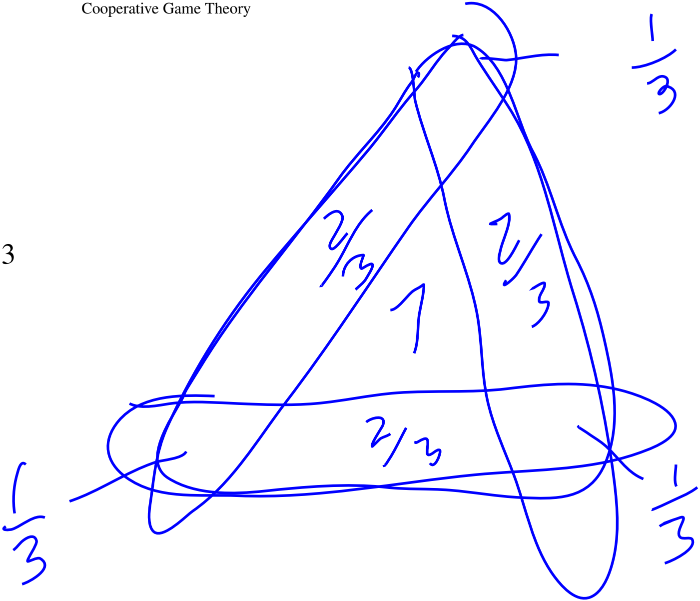
Core empty

- $v(i) = 0$
- $v(i,j) = 0.9$
- $v(N) = 1$



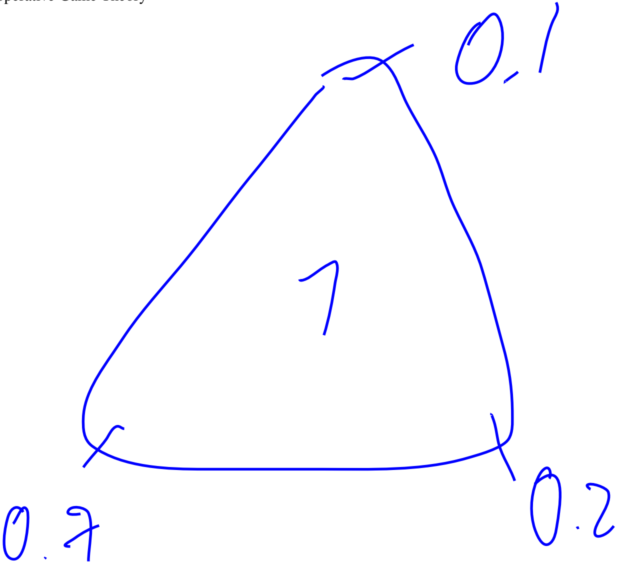
Core unique

- $v(i) = 0$
- $v(i,j) = 2/3$
- $v(N) = 1$



Core large

- $v(i) = v(i,j) = 0$
- $v(N) = 1$



Bondareva-Shapley Theorem

Bondareva 1963 and Shapley 1967

The Core of a cooperative game is *nonempty* if and only if the game is *balanced*.

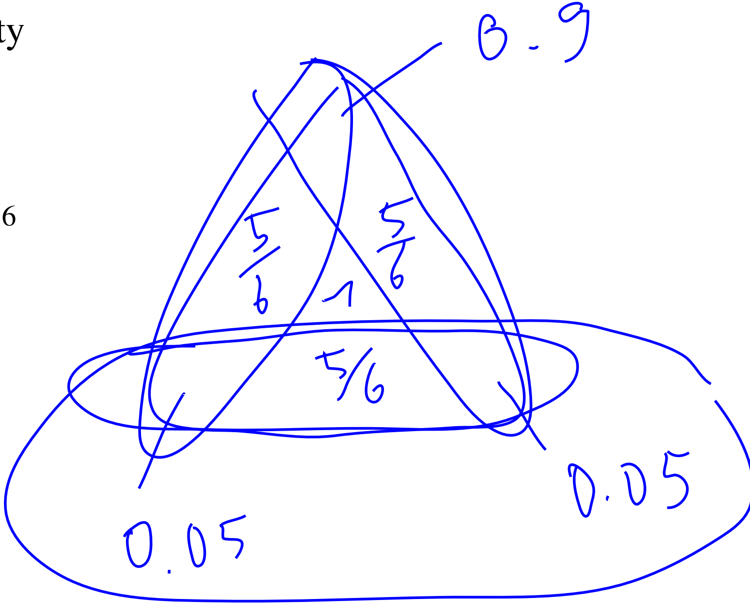
Balancedness:

- *Balancing weight:* Let $\alpha(C) \in [0, 1]$ be the balancing weight attached to any $C \in 2^N$
- *Balanced family:* A set of balancing weights α is a balanced family if, for every i, $\sum_{C \in 2^N : i \in C} \alpha(C) = 1$
- Balancedness in a superadditive game then requires that, for all balanced families,
  - $v(N) \geq \sum_{C \in 2^N} \alpha(C) v(C)$

Limitations of the Core

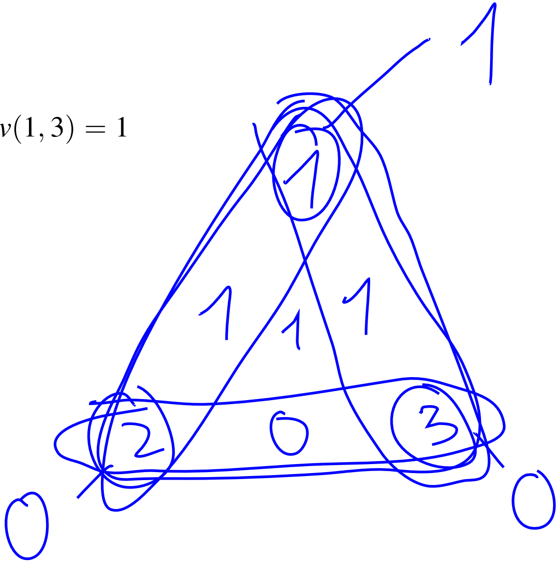
1. Core empty

- $v(i) = 0$
- $v(i,j) = 5/6$
- $v(N) = 1$



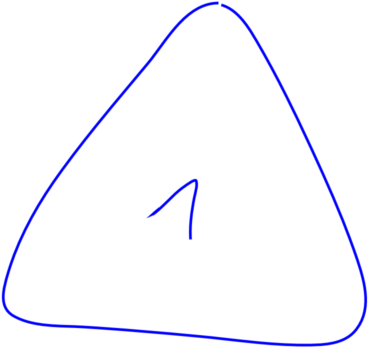
2. Core non-empty but very inequitable (1, 0, 0)

- $v(i) = v(2,3) = 0$
- $v(N) = v(1,2) = v(1,3) = 1$



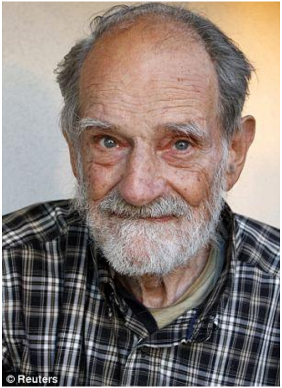
3. Core large (any split of 1)

- $v(i) = v(i,j) = 0$
- $v(N) = 1$



So is the Core a *descriptive* or a *prescriptive/normative* solution concept?

What about an explicitly *normative* solution concept?



Lloyd Shapley (1923-2016)

Shapley value (Shapley 1953)

Axioms. Given some $G(v, N)$, an acceptable allocation/value $x^*(v)$ should satisfy

- **Efficiency.** $\sum_{i \in N} x_i^*(v) = v(N)$
- **Symmetry**. if, for any two players i and j, $v(S \cup i) = v(S \cup j)$ for all S not including i and j, then $x_i^*(v) = x_j^*(v)$
- **Dummy player**. if, for any i, $v(S \cup i) = v(S)$ for all S not including i, then $x_i^*(v) = 0$
- **Additivity**. If u and v are two characteristic functions, then $x^*(v + u) = x^*(v) + x^*(u)$

Shapley's characterization

The unique function satisfying all four axioms for the set of all games is

$$\phi_i(v) = \sum_{S \in N, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \setminus \{i\})]$$

So what does this function mean?

Shapley value

The Shapley value pays each player his *average marginal contributions*:

- For any S: $i \in S$, think of the *marginal contribution* $MC_i(S) = v(S) - v(S \setminus i)$.
- And of $\sum_{S \in N, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}$ as some kind of "average" operator (more detail later).

Then,

$$\phi_i(v) = \text{average}(MC_i(S))$$

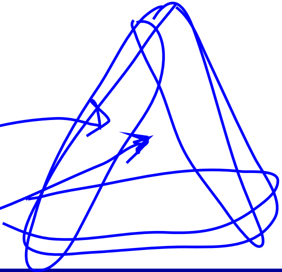
An alternative characterization

Young (1985): a set of equivalent axioms is

- **Efficiency**. $\sum_{i \in N} x_i^*(v) = v(N)$
- **Symmetry**. if, for any two players i and j, $v(S \cup i) = v(S \cup j)$ for all S not including i and j, then $x_i^*(v) = x_j^*(v)$
- **Monotonicity**. If u and v are two characteristic functions and, for all S including i, $u(S) \geq v(S)$, then $x_i^*(u) \geq x^*(v)$

A more attractive set of axioms...

1. Core empty

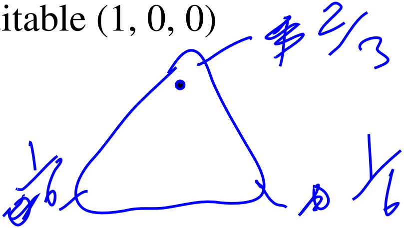


- $v(i) = 0$
- $v(i,j) = 5/6$
- $v(N) = 1$

Shapley value

$(1/3, 1/3, 1/3)$

2. Core non-empty but very inequitable (1, 0, 0)



- $v(i) = v(2,3) = 0$
- $v(N) = v(1,2) = v(1,3) = 1$

Shapley value

$(4/6, 1/6, 1/6)$

3. Core large (any split of 1)

- $v(i) = v(i,j) = 0$
- $v(N) = 1$

Shapley value

$(1/3, 1/3, 1/3)$

Room-entering story (Roth 1983)

Average MC in this sense...

- $N = \{1, 2, ..., n\}$ players enter a room in some order.
- Whenever a player enters a room, and players $S \setminus i$ are already there, he is paid his marginal contribution $MC_i(S) = v(S) - v(S \setminus i)$.
- Suppose all $n!$ orders are equally likely.
- Then there are $(s-1)!$ different orders in which these players in $S \setminus i$ can precede i
- and $(n-s)!$ order in which the others may follow
- hence, a total of $(s-1)!(n-s)!$ orders for that case of the $n!$ total orders.
- Ex ante, the payoff of a players is $\sum_{S \in N, i \in S} \frac{(s-1)!(n-s)!}{n!} MC_i(S)$ – the Shapley value.

Relationship between the Core and the Shapley value

Put simply, none...

- the Shapley value is normative
- the Core is something else (hybrid)
- when the Core is non-empty, the SV may lie inside or not
- when the Core is empty, the SV is still uniquely determined

Other cooperative models

Non-transferable-utility cooperative game

- **As before**: CFG defined by 2-tuple $G(v,N)$
- **Outcome**: *partition* $\rho = \{C_1, C_2, ..., C_k\}$ *directly* (w/o negotiating how to share) implies a payoff allocation/imputation – $\phi_i = f_i(C_i)$
- There are no side-payments and the worth of a coalition cannot be (re-)distributed.

Agents have preferences over coalitions.

Stable Marriage/Matching problem

2-sided market

- *Men* $M = \{m_1, ..., m_n\}$ on one side, *women* $W = \{w_1, ..., w_n\}$ on the other.
- Each m_i: preferences (e.g. $w_1 \succ w_2 \succ ... \succ w_n$) over women
- Each w_i: preferences (e.g. $m_n \succ m_1 \succ ... \succ m_{n-1}$) over men

THANKS EVERYBODY
and keep checking the website for new materials as we progress!
http://gametheory.online