Exercises for "Introduction to Game Theory"
SOLUTIONS

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1 Cooperative game theory

Exercise 1.1

Marginal contributions

1. If the value of coalition (A,B,C) is v(A,B,C)=100, and the value of coalition (A,B) is v(A,B)=30, and the value of C is v(C)=20, what is the marginal contribution of player C to coalition (A,B,C)?
2. If v(A)=20 and v(B)=0, what is the marginal contribution of B to (A,B)?
3. What is the marginal contribution of A to (A,B)?

Answers

1. $v(A, B, C,) - v(A, B) = 70$
2. $v(A, B) - v(A) = 10$
3. $v(A, B) - v(B) = 30$

Exercise 1.2

Superadditivity and the core

1. If the value of coalition N=(A,B,C) is v(N)=100, and the value of the coalition of pairs (i,j) is v(i,j)=30 for all pairs (i,j), and the value of each singleton i is v(i)=0, is the game superadditive?
2. Is the allocation (50,25,25) a core allocation?
3. Is the allocation (30,30,30) a core allocation?

4. Is the allocation (40,10,50) a core allocation?
5. Is the allocation (80,10,10) a core allocation?

Answers

1. yes, adding more players "increases the pie"
2. yes
3. no
4. yes
5. no

Exercise 1.3

Cooperative games

A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

1. Which coalitions of parties can obtain a majority?
2. Is the core non-empty? Can you find a core allocation (if so, write one down)?
3. Calculate the Shapley value.

Answers

1. (A,j) for all j, (A,j,k) for all (j,k), N, and (B,C,D).
2. No, the core is empty. There are many winning coalitions that can block each other.
3. Using the definition of the Shapley value

$$\phi_i(v) = \sum_{S \in N, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \setminus \{i\})]$$

For example for party A we have:

	$(S-1)!$	$(n-\|S\|)!$	$n!$	$v(\cdot) - v(\cdot \backslash \{i\})$	summand
$\{A,B,C,D\}$	6	1	24	$100 - 100$	0
$\{A,B,C\}$	2	1	24	$100 - 0$	100/12
$\{A,B,D\}$	2	1	24	$100 - 0$	100/12
$\{A,C,D\}$	2	1	24	$100 - 0$	100/12
$\{A,B\}$	1	2	24	$100 - 0$	100/12
$\{A,C\}$	1	2	24	$100 - 0$	100/12
$\{A,D\}$	1	2	24	$100 - 0$	100/12
$\{A\}$	1	6	24	$100 - 100$	0

Summing the summands one finds $\phi_A = 50$. Going through the same calculations for all other parties we find: (50, 16.6, 16.6, 16.6)

Exercise 1.4

Cost sharing

Consider three neighboring municipalities, A, B, and C, who can supply themselves with municipal water either by building separate facilities or by building a joint water supply facility.

We suppose that the joint facility is cheaper to construct than the separate projects due to economies of scale. The quantity of water to be supplied to each municipality is assumed given. The problem is then how to divide the costs among them.

We can think of costs as "negative values". Suppose the costs of water supply translate into values such that v(A)=-6.5, v(B)=-4.2, v(C)=-1.5, v(A,B)=-10.3, v(B,C)=-5.3, v(A,C)=-8.0, v(A,B,C)=-10.6.

1. Is the game superadditive?
2. Is the core non-empty? Can you find a core allocation (if so, write one down)?
3. Calculate the Shapley value.

Answers

1. yes
2. yes. e.g. (-6.1, -3.4, -1.1).
3. (-6.033, -3.533, -1.033)

2 Preferences and utilities

Exercise 2.1

Suppose a decision maker is facing a choice over a finite set X and he has the binary preference relation $\succeq$ over X.

1. Which of the following is true? (possibly several)
   (a) If $\succeq$ is transitive there exists a utility function for $\succeq$
   (b) If $\succeq$ is transitive, complete, and satisfies independence of irrelevant alternatives there exists a utility function for $\succeq$
   (c) If $\succeq$ is transitive and complete there exists a utility function for $\succeq$.
2. Give the definition of a utility function for a preference ordering $\succeq$

Answers

1. (a) False

   (b) True

   (c) True

2. ... (see lecture notes)

Exercise 2.2

1. Define the Bernoulli function for preferences $\succ$ on X representing a decision maker's preferences over lotteries over a finite set T.
2. Further define the associated von Neumann-Morgenstern utility function.
3. When does a von Neumann-Morgenstern utility function exist for a preference relation $\succ$

Answers

see lecture notes

Exercise 2.3

Suppose Tic, Tac, and Toe play the following simultaneous game. They each decide whether or not to go to the playground.

- if an odd number turns up at the playground Tic wins
- if exactly two people turn up at the playground Tac wins
- if nobody turns up at the playground Toe wins

Suppose that each player prefers winning to loosing and is indifferent between any two outcomes where he is loosing.

Formulate this game in a normal form with utility functions taking values 0 or 1.

Answers

Let T be turn up and N be not turn up. Tic is the row player, Tac the column player, and Toe chooses the matrix where the

T

	T	N
T	$1,0,0$	$0,1,0$
N	$0,1,0$	$1,0,0$

N

	T	N
T	$0,1,0$	$1,0,0$
N	$1,0,0$	$0,0,1$

3 Non-cooperative game theory

Exercise 3.1

Consider the two-player game with normal form:

	L	R
T	$7, 6$	$0, 5$
B	$2, 0$	$4, 3$

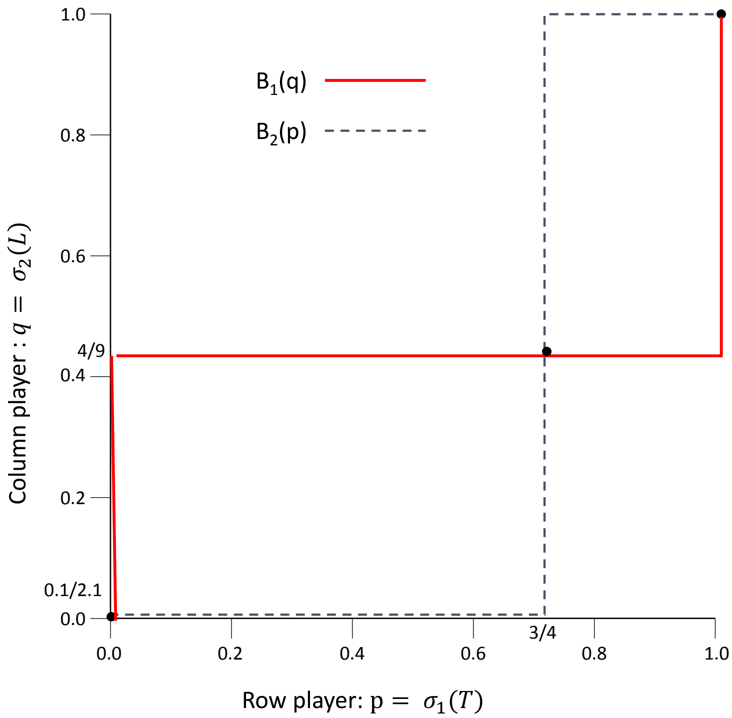
1. Find all Nash equilibria (in pure and mixed strategies)
2. Draw the best-reply graph
3. Find the expected payoff for row and column player in each of the equilibria

Answers

The best responses are underlined:

	L	R
T	$\underline{7}, \underline{6}$	$0, 5$
B	$2, 0$	$\underline{4}, \underline{3}$

1. The pure Nash equilibria are T, L and B, R. To compute the mixed Nash equilibrium suppose that the column player plays L with probability q and R with probability $1 - q$. Then the row player is indifferent if $7q = 2q + 4(1 - q)$. That is $q = 4/9$. Next suppose that the row player plays T with probability p and B with probability $1 - p$. Then the column player is indifferent if $6p = 5p + 3(1 - p)$. That is $p = 3/4$. Thus the unique mixed Nash equilibrium is $\frac{3}{4}T + \frac{1}{4}B, \frac{4}{9}L + \frac{5}{9}R$.
2. The best-reply graph is:



3. (a) T, L: $u_{row} = 7, u_{column} = 6$

   (b) B, R: $u_{row} = 4, u_{column} = 3$

   (c) $\frac{3}{4}T + \frac{1}{4}B, \frac{4}{9}L + \frac{5}{9}R$: $u_{row} = 28/9, u_{column} = 9/2$

Exercise 3.2

Consider the two-player game with normal form:

	l	m	r
T	14, 7	2, 7	2, 0
M	14, 1	10, 5	0, 2
B	0, 1	4, 0	12, 0

1. Find all Nash equilibria (in pure and mixed strategies)
2. Find all pure-strategy perfect equilibria
3. Find the set of iteratively strictly dominated pure strategies for each player
4. Delete all iteratively strictly dominated pure strategies and do tasks 1-3 for the new game

Answers

The best responses are underlined:

	l	m	r
T	$\underline{14}, \underline{7}$	$2, \underline{7}$	$2, 0$
M	$\underline{14}, 1$	$\underline{10}, \underline{5}$	$0, 2$
B	$0, \underline{1}$	$4, 0$	$\underline{12}, 0$

1. The pure Nash equilibria are given by T, l and M, m. For the mixed strategy equilibrium first note that for the column player r is strictly dominated by the mixed strategy $\frac{1}{2}l + \frac{1}{2}m$. Thus r cannot be part of a mixed equilibrium. A similar argument holds for the row player and his strategy B by iterative dominance. Thus consider the case where column is mixing $q \cdot l + (1 - q) \cdot m$. Row player is indifferent only if $q = 1$ and otherwise wants to play M. Next, suppose that row is playing $p \cdot T + (1 - p) \cdot M$. Column is indifferent only if $p = 1$. Thus there are no mixed Nash equilibria.

2. There are two candidates:

   (a) T, l: Suppose that the row player has a trembling hand and plays strategy M with a small probability ε. Then m is the unique best response for the column player. Hence any strategy where column puts probability greater than ε on l is not a perfect equilibrium, this holds in particular for the pure strategy l. Hence T, l is not perfect.

   (b) M, m: By the existence theorem we know that there exists at least one perfect equilibrium. Hence M, m must be perfect.

3. see 1.

4. see above.

Exercise 3.3

Consider the following application. Bonnie and Clyde are to divide their latest robbery of 1 unit of gold. Each of them has (von Neumann-Morgenstern) utility $u(g) = \sqrt{(g)}$ from receiving $g \in [0, 1]$ share of the gold.

1. Suppose each of them has to submit a suggestion for her share, say $b \in [0, 1]$ for Bonnie and $c \in [0, 1]$ for Clyde. The suggestions are written down independently. If the suggestions are compatible, that is, $b + c \leq 1$, then they each get their suggested share, otherwise both get nothing.

   (a) Define this as a game

   (b) Find the set of pure strategy Nash equilibria

2. Now suppose that Bonnie submits her suggestion before Clyde and suppose that Clyde hears about Bonnie's suggestion before she makes hers.

   (a) Define this as a game

   (b) Find the set of pure strategy subgame-perfect equilibria

   (c) Give an example of a Nash equilibrium that is not subgame-perfect

3. Suppose Bonnie and Clyde are able to write binding contracts. Find the Shapley value of the game

Answers

1. (a) The set of player is $N = \{Bonnie, Clyde\}$. The strategy set is the same for both players $S = [0, 1]$. The utility function for Bonnie is given by

$$u_{Bonnie}(b, c) = \sqrt{b} \quad \text{if} \quad b + c \leq 1$$
$$u_{Bonnie}(b, c) = 0 \quad \text{else}$$

   Similarly, the utility function for Clyde is given by

$$u_{Clyde}(b, c) = \sqrt{c} \quad \text{if} \quad b + c \leq 1$$
$$u_{Clyde}(b, c) = 0 \quad \text{else}$$

   (b) The set of pure strategy Nash equilibria consists of any tupple (b, c) such that $b + c = 1$ (and $b \geq 0, c \geq 0$) and the tupple (b, c) where $b = c = 1$.

2. Now suppose that Bonnie submits her suggestion before Clyde and suppose that Clyde hears about Bonnie's suggestion before she makes hers.

   (a) Players and utilities are as before. The strategies for Bonnie are again $b \in [0, 1]$. The strategies for Clyde are given by a function f mapping $b \to c \in [0, 1]$.

   (b) The unique best response for Clyde is given by $1 - b$ if $b < 1$. If $b = 1$ any strategy is a best response for Clyde. Hence, knowing this Bonnie will play $b < 1$ maximal to enforce a certain equilibrium giving her the maximal payoff.

   (c) For example $b = 0.5, c = 0.5$ is not subgame perfect. Bonnie, by, for example, playing $b = 0.7$ can do better.

3. Using the definition of the Shapley value

$$\phi_i(v) = \sum_{S \in N, i \in S} \frac{(|S| - 1)!(n - |S|)!}{n!}[v(S) - v(S \setminus \{i\})]$$

   we find: $1/2 \cdot 0 + 1/2 \cdot 0 + 1/2 \cdot 1 = 1/2$ (Note: you need to assume that each singleton coalition yields 0.)

Exercise 3.4

Suppose a player has a strictly dominant strategy in a normal form game.

1. Is the following statement true "She is sure to get her best possible outcome in any Nash equilibrium of the game"?
2. Explain your answer and give an example of a game that illustrates your answer.

Answers

1. The statement is not true.
2. Nash equilibrium is a stability concept and not an efficiency concept. While a strategy is may be strictly dominant the outcome which is best for all players may still involve dominated strategies. A prominent example is the Prisoner's dilemma.

Exercise 3.5

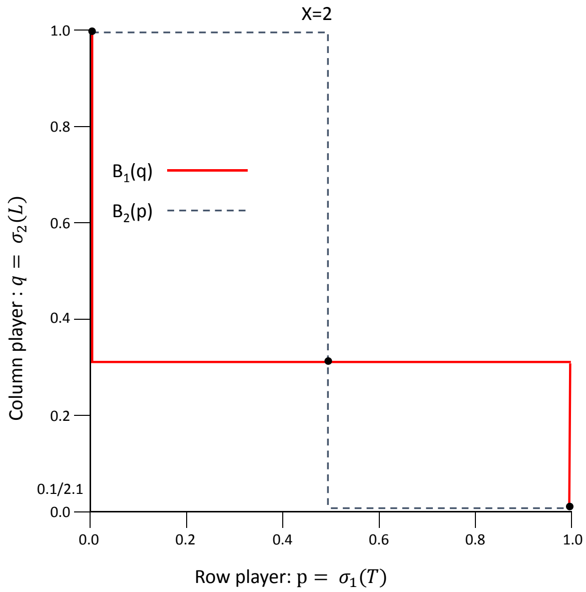
Consider the two-player game with normal form:

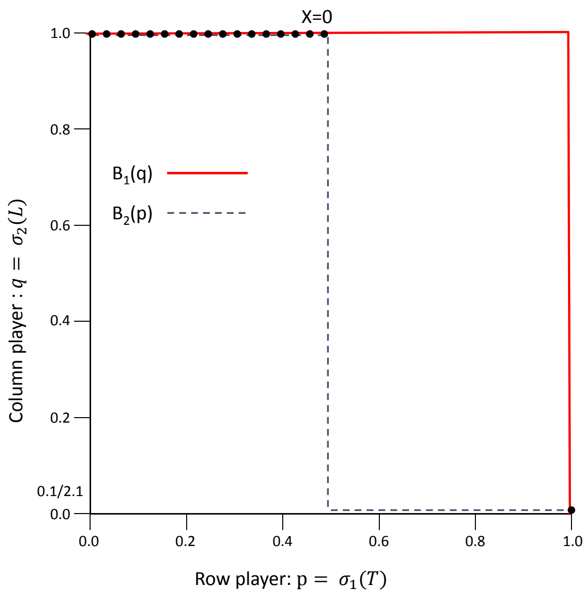
	L	R
T	$0, 0$	$-1, 1$
B	$X, -1$	$-2, -2$

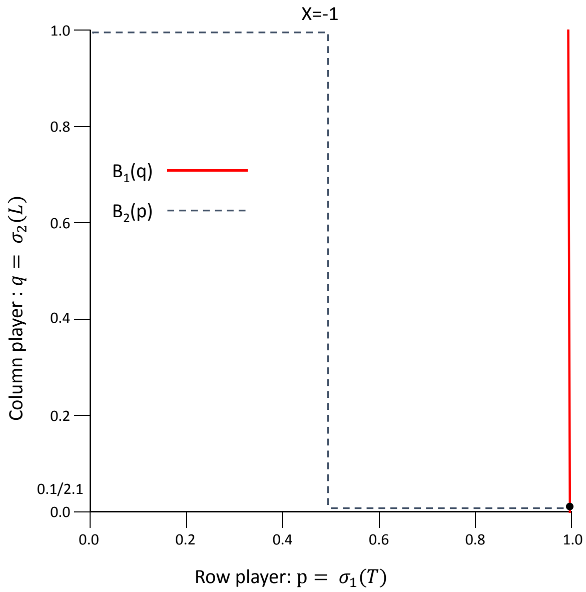
1. Find all pure and mixed strategy Nash equilibria if $X > 0$
2. Find all pure and mixed strategy Nash equilibria if $X < 0$
3. If $X = 2$ is row player more likely to play T or column player more likely to play L? Which player gets the higher expected payoff.
4. Draw the best-reply graph for $X = 2, X = 0, X = -1$

Answers

1. $(B, L), (T, R)$ and the mixed equilibrium where row player plays T with probability 0.5 and column player plays L with probability $1/(1 + X)$
2. (T, R)
3. Row player is more likely to play T than column player is likely to play L. Row player's expected payoff is $-2/3$, column player's expected payoff is $-1/2$
4. The best reply graphs are:

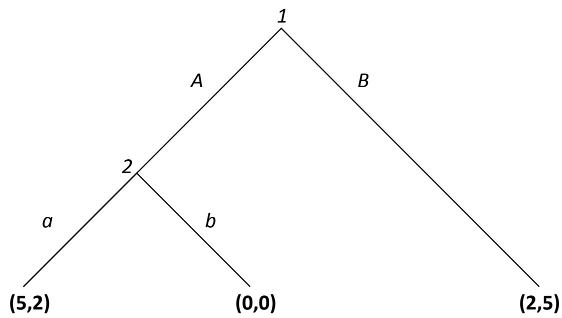

X=2
B1(q)
B2(p)
Column player : q = σ2(L)
Row player: p = σ1(T)
1.0
0.8
0.6
0.4
0.2
0.1/2.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0


X=0
B1(q)
B2(p)
Column player : q = σ2(L)
Row player: p = σ1(T)
1.0
0.8
0.6
0.4
0.2
0.1/2.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0



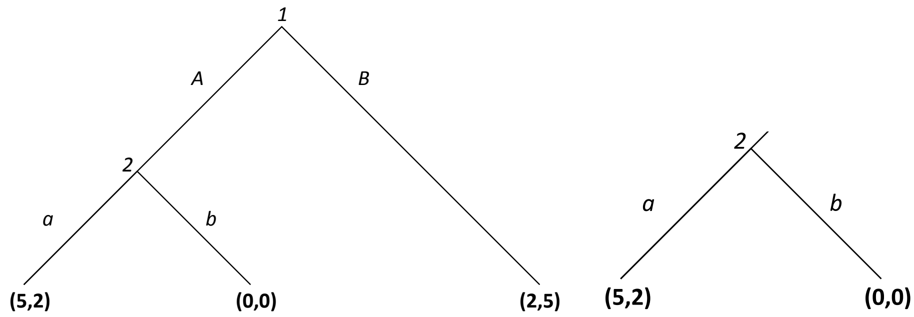
Exercise 3.6

Given the following extensive form game:



1. Write down all subgames
2. Identify the pure strategy sets for both players
3. Write down the normal form representation of the game with player 1 as row player and player 2 as column player
4. Find the set of Nash equilibria of the game
5. Write down the definition of subgame perfect equilibrium
6. Find the unique subgame perfect equilibrium of the game

Answers

1. 

2. Player 1: A, B; player 2: a, b

3. Normal form representation

	a	b
A	$5, 2$	$0, 0$
B	$2, 5$	$2, 5$

4. The best responses are underlined

	a	b
A	$\underline{5}, \underline{2}$	$0, 0$
B	$2, \underline{5}$	$\underline{2}, \underline{5}$

5. ... see lecture notes

6. Player 2 plays a in the subgame. Thus player 1 plays A. The unique subgame equilibrium yields payoffs $5, 2$.