

# COOPERATIVE GAME THEORY: CORE AND SHAPLEY VALUE

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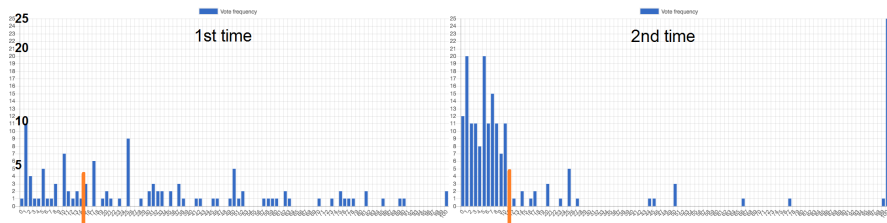
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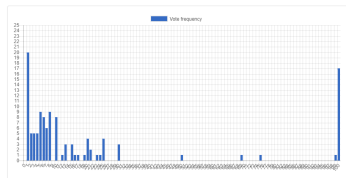
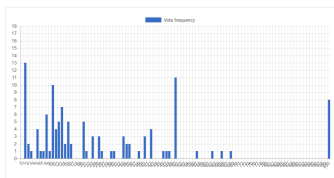
Eidgenössische Technische Hochschule Zürich  
Swiss Federal Institute of Technology Zurich

# What you did last week...



There appear to be traces of both, more strategic behavior (more mass around 0 and 1) and more collusive/cooperative behavior (the 100s), when you played the game the second time.

Actually, that was the 2017 class... you did:



There appear to be traces of both, more strategic behavior (even more mass around 0 and 1) but also more collusive/cooperative behavior (the 100s), when you played the game the second time.

# The **two** branches of game theory

## *Non-cooperative* game theory

- No binding contracts can be written
- Players are individuals
- Nash equilibrium

## *Cooperative* game theory

- Binding contract can be written
- Players are individuals and coalitions of individuals
- Main solution concepts:
  - Core
  - Shapley value
- **The focus of today!**

## Reminder: the ingredients of a **noncooperative game**

- **Players:**  $N = \{1, 2, \dots, n\}$
- **Actions / strategies:** each player chooses  $s_i$  from his own finite strategy set;  $S_i$  for each  $i \in N$ 
  - **Outcome:** resulting strategy combination:  $s = (s_1, \dots, s_n) \in (S_i)_{i \in N}$
- **Payoff outcome:** payoffs  $u_i = u_i(s)$

# The Theory of Games and Economic Behavior (1944)



John von Neumann (1903-1957) and Oskar Morgenstern (1902-1977)

# The ingredients of a **cooperative game**

- **Population of players:**  $N = \{1, 2, \dots, n\}$  (finite)
- **Coalitions:**  $C \subseteq N$  form in the population and become players
  - resulting in a coalition structure  $\rho = \{C_1, C_2, \dots, C_k\}$
- **Payoffs:** *–we still need to specify how–* payoffs  $\phi = \{\phi_1, \dots, \phi_n\}$  come about:
  - something like this:  $\phi_i = \phi(\rho, \text{“sharing rule”})$

# Cooperative games in **characteristic function form** (CFG)

- **The game:** A CFG defined by 2-tuple  $G(v, N)$
- **Players:**  $N = 1, 2, \dots, n$  (finite, fixed population)
- **Coalitions:** *disjoint*  $C \subseteq N$  form resulting in a coalition structure/  
partition  $\rho$ 
  - $\emptyset$  is an *empty coalition*
  - $N$  is the *grand coalition*
  - The set of all coalitions is  $2^N$
  - $\rho$  is the set of all partitions
- **Characteristic function:**  $v$  is the characteristic function form that assigns a *worth*  $v(C)$  to each coalition
  - $v: 2^N \rightarrow R$
  - (and  $v(\emptyset) = 0$ )



## 3-player example

- $N=1,2,3$
- $v(i)=0$
- $v(1,2)=v(1,3)=0.5$
- $v(2,3)=0$
- $v(N)=1$

## “Transferable utility” and **feasibility**

- **The game:** CFG defined by 2-tuple  $G(v, N)$
- **Outcome:** *Coalition structure*
  - *partition*  $\rho = \{C_1, C_2, \dots, C_k\}$  and
  - *payoff allocation/imputation*  $\phi = \{\phi_1, \dots, \phi_n\}$
- Importantly,  $v(C)$  can be “shared” amongst  $i \in C$  (**transfer of utils**)!
- **Feasibility:** in each  $C$ ,  $\sum_{i \in C} \phi_i \leq v(C)$

## 3-player example: some feasible outcomes

- **Outcome 1:**  $\{(1,2),3\}$  and  $\{(0.25,0.25),0\}$
- **Outcome 2:**  $\{N\}$  and  $\{0.25,0.25,0.5\}$
- **Outcome 3:**  $\{N\}$  and  $\{0.8,0.1,0.1\}$

# Superadditivity assumption

## Superadditivity

If two coalitions  $C$  and  $S$  are disjoint (i.e.  $S \cap C = \emptyset$ ), then  $v(C) + v(S) \leq v(C \cup S)$

i.e. “mergers of coalitions weakly improve their worths”

- Superadditivity implies *efficiency* of the grand coalition: for all  $\rho \in \rho$ ,  $v(N) \geq \sum_{C \in \rho} v(C)$ .
- In our example:  
 $v(N) > v(1, 2) = v(1, 3) > v(2, 3) = v(1) = v(2) = v(3)$ .

# The Core (Gillies 1959)

## The Core

*The Core* of a superadditive  $G(v, n)$  consists of all outcomes where the *grand coalition* forms and payoff allocations  $\phi^*$  are

Pareto-efficient:  $\sum_{i \in N} \phi_i^* = v(N)$

Unblockable: for all  $C \subset N$ ,  $\sum_{i \in C} \phi_i^* \geq v(C)$

- *individual rational*:  $\phi_i^* \geq v(i)$  for all  $i$
- *coalitional rational*:  $\sum_{i \in C} \phi_i^* \geq v(C)$  for all  $C$

## 3-player example

- **Outcome 1:**  $\{(1,2),3\}$  and  $\{(0.25,0.25),0\}$
- **Outcome 2:**  $\{N\}$  and  $\{0.25,0.25,0.5\}$
- **Outcome 3:**  $\{N\}$  and  $\{0.8,0.1,0.1\}$

# Properties of the Core

- A system of weak **linear** inequalities defines the Core, which is therefore **closed** and **convex**.
- The core can be
  - **empty**
  - **non-empty**
  - **large**

# Core empty

- $v(i) = 0$
- $v(i, j) = 0.9$
- $v(N) = 1$



# Core unique

- $v(i) = 0$
- $v(i, j) = 2/3$
- $v(N) = 1$

# Core large

- $v(i) = v(i,j) = 0$
- $v(N) = 1$

# Bondareva-Shapley Theorem

## Bondareva 1963 and Shapley 1967

The Core of a cooperative game is *nonempty* **if and only if** the game is *balanced*.

### Balancedness:

- *Balancing weight*: Let  $\alpha(C) \in [0, 1]$  be the balancing weight attached to any  $C \in 2^N$
- *Balanced family*: A set of balancing weights  $\alpha$  is a balanced family if, for every  $i$ ,  $\sum_{C \in 2^N: i \in C} \alpha(C) = 1$
- Balancedness in a superadditive game then requires that, for all balanced families,
  - $v(N) \geq \sum_{C \in 2^N} \alpha(C)v(C)$

# Limitations of the Core

# 1. Core empty

- $v(i) = 0$
- $v(i, j) = 5/6$
- $v(N) = 1$

## 2. Core non-empty but very inequitable $(1, 0, 0)$

- $v(i) = v(2, 3) = 0$
- $v(N) = v(1, 2) = v(1, 3) = 1$

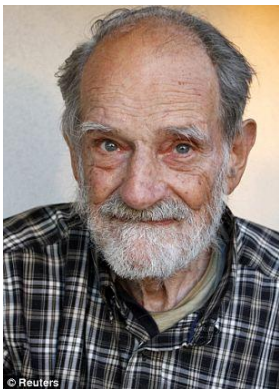
### 3. Core large (any split of 1)

- $v(i) = v(i,j) = 0$
- $v(N) = 1$

So is the Core a *descriptive* or a *prescriptive/normative* solution concept?



What about an explicitly *normative* solution concept?



Lloyd Shapley (1923-2016)

# Shapley value (Shapley 1953)

**Axioms.** Given some  $G(v, N)$ , an acceptable allocation/value  $x^*(v)$  should satisfy

- **Efficiency.**  $\sum_{i \in N} x_i^*(v) = v(N)$
- **Symmetry.** if, for any two players  $i$  and  $j$ ,  $v(S \cup i) = v(S \cup j)$  for all  $S$  not including  $i$  and  $j$ , then  $x_i^*(v) = x_j^*(v)$
- **Dummy player.** if, for any  $i$ ,  $v(S \cup i) = v(S)$  for all  $S$  not including  $i$ , then  $x_i^*(v) = 0$
- **Additivity.** If  $u$  and  $v$  are two characteristic functions, then  $x^*(v + u) = x^*(v) + x^*(u)$

# Shapley's characterization

The unique function satisfying all four axioms for the set of all games is

$$\phi_i(v) = \sum_{S \in \mathcal{N}, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!} [v(S) - v(S \setminus \{i\})]$$

**So what does this function mean?**

# Shapley value

The Shapley value pays each player his *average marginal contributions*:

- For any  $S: i \in S$ , think of the *marginal contribution*  
 $MC_i(S) = v(S) - v(S \setminus i)$ .
- And of  $\sum_{S \in \mathcal{N}, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}$  as some kind of “average” operator (more detail later).

Then,

$$\phi_i(v) = \text{average} (MC_i(S))$$

## An alternative characterization

### Young (1985): a set of equivalent axioms is

- **Efficiency.**  $\sum_{i \in N} x_i^*(v) = v(N)$
- **Symmetry.** if, for any two players  $i$  and  $j$ ,  $v(S \cup i) = v(S \cup j)$  for all  $S$  not including  $i$  and  $j$ , then  $x_i^*(v) = x_j^*(v)$
- **Monotonicity.** If  $u$  and  $v$  are two characteristic functions and, for all  $S$  including  $i$ ,  $u(S) \geq v(S)$ , then  $x_i^*(u) \geq x_i^*(v)$

**A more attractive set of axioms...**

# 1. Core empty

- $v(i) = 0$
- $v(i, j) = 5/6$
- $v(N) = 1$

## Shapley value

$(1/3, 1/3, 1/3)$

## 2. Core non-empty but very inequitable (1, 0, 0)

- $v(i) = v(2, 3) = 0$
- $v(N) = v(1, 2) = v(1, 3) = 1$

### Shapley value

$(4/6, 1/6, 1/6)$



### 3. Core large (any split of 1)

- $v(i) = v(i,j) = 0$
- $v(N) = 1$

#### Shapley value

$(1/3, 1/3, 1/3)$

## Room-entering story (Roth 1983)

### Average MC in this sense...

- $N = \{1, 2, \dots, n\}$  players enter a room in some order.
- Whenever a player enters a room, and players  $S \setminus i$  are already there, he is paid his marginal contribution  $MC_i(S) = v(S) - v(S \setminus i)$ .
- Suppose all  $n!$  orders are equally likely.
- Then there are  $(s - 1)!$  different orders in which these players in  $S \setminus i$  can precede  $i$
- and  $(n - s)!$  order in which the others may follow
- hence, a total of  $(s - 1)!(n - s)!$  orders for that case of the  $n!$  total orders.
- Ex ante, the payoff of a players is  $\sum_{S \in N, i \in S} \frac{(s-1)!(n-s)!}{n!} MC_i(S) -$  the Shapley value.

## Relationship between the Core and the Shapley value

Put simply, none...

- the Shapley value is normative
- the Core is something else (hybrid)
- when the Core is non-empty, the SV may lie inside or not
- when the Core is empty, the SV is still uniquely determined

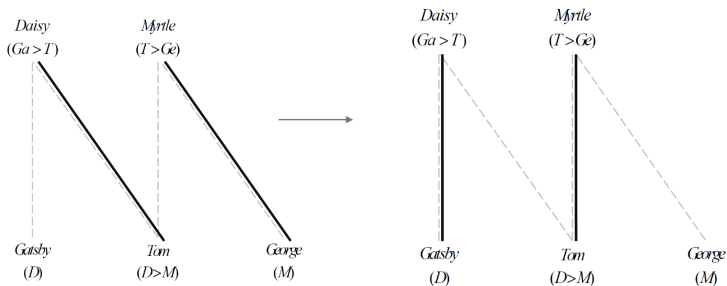
# Other cooperative models

## Non-transferable-utility cooperative game

- **As before:** CFG defined by 2-tuple  $G(v, N)$
- **Outcome:** *partition*  $\rho = \{C_1, C_2, \dots, C_k\}$  *directly* (w/o negotiating how to share) implies a payoff allocation/imputation –  $\phi_i = f_i(C_i)$
- There are no side-payments and the worth of a coalition cannot be (re-)distributed.

*Agents have preferences over coalitions.*

# Stable Marriage/Matching problem



# Stable Marriage/Matching problem

## 2-sided market

- Men  $M = \{m_1, \dots, m_n\}$  on one side, women  $W = \{w_1, \dots, w_n\}$  on the other.
- Each  $m_i$ : preferences (e.g.  $w_1 \succ w_2 \succ \dots \succ w_n$ ) over women
- Each  $w_i$ : preferences (e.g.  $m_n \succ m_1 \succ \dots \succ m_{n-1}$ ) over men

**We want to establish a stable matching:** forming couples (man-woman) such that there exists no alternative couple where both partners prefer to be matched with each other rather than with their current partners.

# Deferred acceptance

## Gale-Shapley 1962

*For any marriage problem, one can make all matchings stable using the deferred acceptance algorithm.*

Widely used in practice (e.g. Roth & Sotomayor 1990, Roth et al. ...):

- Resource allocations/doctor recruitment for hospitals
- Organ transplantations
- School admissions/room allocation
- Assigning users to servers in distributed Internet services
- ...



## DA “pseudo-code”

**Initialize** : all  $m_i \in M$  and all  $w_i \in W$  are *single*.

**Engage** : Each single man  $m \in M$  *proposes* to his *preferred* woman  $w$  to whom he has *not yet proposed*.

- If  $w$  is single, she will become *engaged* with her *preferred proposer*.
- Else  $w$  is already engaged with  $m'$ .
  - If  $w$  prefers her preferred proposer  $m$  over her current engagement  $m'$ , then  $(m, w)$  become engaged and  $m'$  becomes single.
  - Else  $(m', w)$  remain engaged.
- All proposers who do not become engaged remain single.

**Repeat** : If there exists a single man after **Engage**, repeat **Engage**; Else move to **Terminate**.

**Terminate** : *Marry* all engagements.

## Proof sketch

**Trade up** : Women can *trade up* until every woman (hence also every man) is engaged, which is when they all get married.

**Termination** : No singles can remain, because every man would eventually propose to every woman as long as he remains single, and every single woman, once proposed to, becomes engaged.

**Termination with stability?**

# Proof sketch

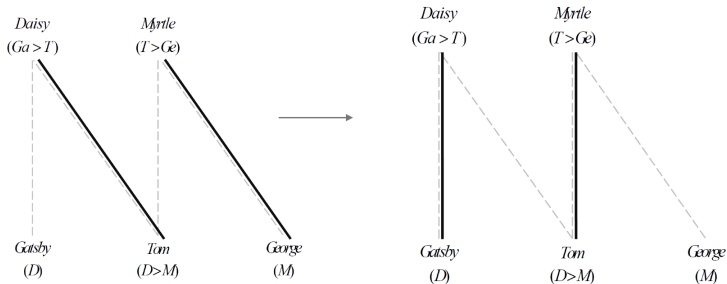
**Stability** : *The resulting matching is stable.*

*Proof* : Suppose the algorithm terminates so that there exists a pair  $(m, w)$  whose partners are engaged to  $w' \neq w$  and  $m' \neq m$  respectively.

**Claim** : It is not possible for both  $m$  and  $w$  to prefer each other over their engaged partner. because

- If  $m$  prefers  $w$  over  $w'$ , then he proposed to  $w$  before he proposed to  $w'$ . At that time,
  - *Case 1:* If  $w$  got engaged with  $m$ , but did not marry him, then  $w$  must have traded up and left  $m$  for someone she prefers over  $m$ , and therefore cannot prefer  $m$  over  $m'$ .
  - *Case 2:* Else, if  $w$  did not get engaged with  $m$ , then she was already with someone she prefers to  $m$  at that time, and can therefore not prefer  $m$  over  $m'$ .
- Hence, either  $m$  prefers  $w'$  over  $w$ , or  $w$  prefers  $m'$  over  $m$ .

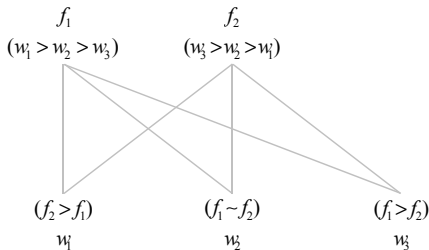
# Back to the Great Gatsby



## From NTU to TU matching

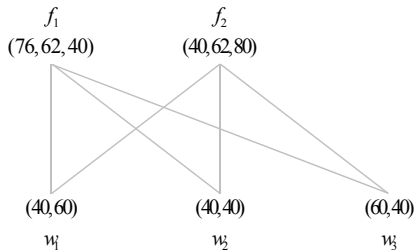
### Non-transferable utility

*Pairwise stable* outcomes always exist  
(Gale & Shapley 1962)



### Transferable utility

*Pairwise stable* and *optimal* outcomes  
(core) always exist (Shapley & Shubik  
1972)



## Firms/workers and their willingness to pay/accept

Firms  $i \in F$  and workers  $j \in W$  look for partners ( $|F| = |W| = N$ )

- Firm  $i$  is willing to pay at most  $r_i^+(j) \in \delta\mathbb{N}$  to match worker  $j$
- Worker  $j$  is willing to accept at least  $r_j^-(i) \in \delta\mathbb{N}$  to match firm  $i$

$\delta > 0$  is the minimum unit (‘dollars’)

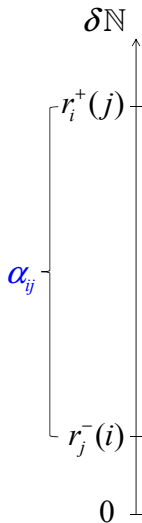


## The resulting match values $\alpha$

The match value for the pair  $(i, j)$  is

$$\alpha_{ij} = (r_i^+(j) - r_j^-(i))_+$$

Define matrix  $\alpha = (\alpha_{ij})_{i \in F, j \in W}$



## Match values and the assignment matrix define the game

Let  $\mathbf{A} = (a_{ij})_{i \in F, j \in W}$  be the **assignment** such that *each agent has at most one partner* and

$$\text{if } (i, j) \text{ is } \begin{cases} \text{matched} & \text{then } a_{ij} = 1 \\ \text{unmatched} & \text{then } a_{ij} = 0 \end{cases}$$



## Prices and payoffs

**Price.** Any price  $\pi_{ij}$  between the willingnesses to “buy” and “sell” of firm and worker is individual rational:

$$r_i^+(j) \geq \pi_{ij} \geq r_j^-(i)$$

**Payoff.** Given prices, payoffs are

The **payoff** to firm  $i$  is  $\phi_i = r_i^+(j) - \pi_{ij}$

The **payoff** to worker  $j$  is  $\phi_j = \pi_{ij} - r_j^-(i)$

If an agent  $i$  (no matter whether firm or worker) is single  $\phi_i = 0$ .

**Assignment  $\mathbf{A}$**  and **payoffs  $\phi$**  define an **outcome** of the game.

## Solution concepts

**Optimality.**  $\mathbf{A}$  is *optimal* if it maximizes total payoff:

$$\sum_{(i,j) \in F \times W} a_{ij} \cdot \alpha_{ij}$$

**Pairwise stability.**  $\phi$  is *pairwise stable* if for all  $(i,j)$  matched

$$\phi_i + \phi_j = \alpha_{ij}$$

and for all  $k, l$  not matched

$$\phi_k + \phi_l \geq \alpha_{kl}$$

**Core.** The *Core* of an assignment game consists of all outcomes,  $[\mathbf{A}, \phi]$ , such that  $\mathbf{A}$  is an optimal assignment and  $\phi$  is pairwise stable.

### Shapley-Shubik 1972

*The Core of the assignment game is nonempty.*

## Example

Shapley-Shubik's **house-trading game**:

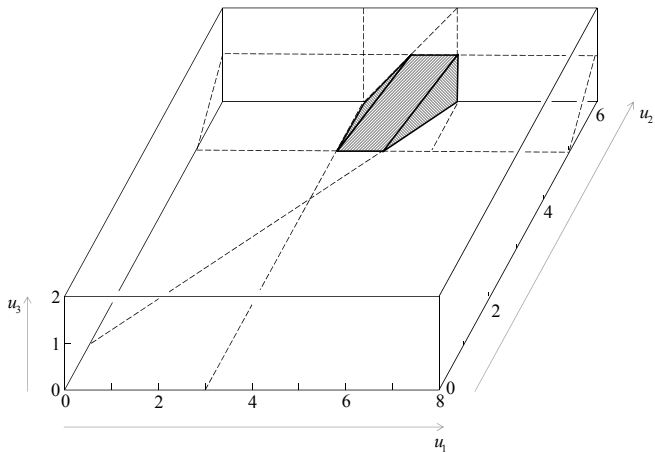
| House | Sellers willingness to accept    | Buyers' willingness to pay |            |            |
|-------|----------------------------------|----------------------------|------------|------------|
|       | $q_{1j}^+ = q_{2j}^+ = q_{3j}^+$ | $p_{i1}^+$                 | $p_{i2}^+$ | $p_{i3}^+$ |
| 1     | 18,000                           | 23,000                     | 26,000     | 20,000     |
| 2     | 15,000                           | 22,000                     | 24,000     | 21,000     |
| 3     | 19,000                           | 21,000                     | 22,000     | 17,000     |

These prices lead to the following match values,  $\alpha_{ij}$  (units of 1000), where sellers are occupying rows and buyers columns:

$$\alpha = \begin{pmatrix} 5 & \mathbf{8} & 2 \\ 7 & 9 & \mathbf{6} \\ \mathbf{2} & 3 & 0 \end{pmatrix}$$

Unique optimal matching is shown in **bold** numbers.

Figure: Core imputation space for the sellers.



## How to “implement” a core outcome?

**Centralized market:** Algorithm by central planner yields stable and optimal outcomes, e.g.,

- *deferred acceptance* (Gale & Shapley 1962) for the NTU game
- solving a LP-dual (Shapley & Shubik 1972) for the TU game
- used in practice: hospital matching, transplantations, school admissions, internet servers, communication networks, etc.

**Decentralized market:** Players trade and (re-)match repeatedly over time.

- random *blocking paths* (Roth & Vande Vate 1990) converge to the NTU core
- random *(re-)trade and price adjustment* (Nax & Pradelski 2015) converge to the TU core
- used in practice: “free” markets, internet auctions, labor market, etc.

**THANKS EVERYBODY**

Bary will take over for a few lectures next week!

and keep checking the website for new materials as we progress:

<http://www.coss.ethz.ch/education/GT.html>

<http://gametheory.online>