NORMAL FORM GAMES: invariance and refinements
DYNAMIC GAMES: extensive form

Heinrich H. Nax
hnax@ethz.ch

Bary S. R. Pradelski
bpradelski@ethz.ch

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Plan

- **Normal form games**
  - Equilibrium invariance
  - Equilibrium refinements

- **Dynamic games**
  - Extensive form games
  - Incomplete information
  - Sub-game perfection
Nash’s equilibrium existence theorem

**Theorem (Nash 1951)**

Every finite game has at least one [Nash] equilibrium in mixed strategies.
Cook book: How to find mixed Nash equilibria

- Find all pure strategy NE.

Check whether there is an equilibrium in which row mixes between several of her strategies:

- Identify candidates:
  - If there is such an equilibrium then each of these strategies must yield the same expected payoff given column’s equilibrium strategy.
  - Write down these payoffs and solve for column’s equilibrium mix.
  - Reverse: Look at the strategies that column is mixing on and solve for row’s equilibrium mix.

- Check candidates:
  - The equilibrium mix we found must indeed involve the strategies for row we started with.
  - All probabilities we found must indeed be probabilities (between 0 and 1).
  - Neither player has a positive deviation.
Battle of the Sexes revisited

**Players** The players are the two students \( N = \{\text{row, column}\} \).

**Strategies** Row chooses from \( S_{\text{row}} = \{\text{Cafe, Pub}\} \)
Column chooses from \( S_{\text{column}} = \{\text{Cafe, Pub}\} \).

**Payoffs** For example, \( u_{\text{row}}(\text{Cafe, Cafe}) = 4 \). The following matrix summarises:

<table>
<thead>
<tr>
<th></th>
<th>Cafe (q)</th>
<th>Pub (1 - q)</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe (p)</td>
<td>4, 3</td>
<td>1, 1</td>
<td>(4q + (1 - q))</td>
</tr>
<tr>
<td>Pub (1 - p)</td>
<td>0, 0</td>
<td>3, 4</td>
<td>(3(1 - q))</td>
</tr>
<tr>
<td>Expected</td>
<td>3(p)</td>
<td>(p + 4(1 - p))</td>
<td></td>
</tr>
</tbody>
</table>

Column chooses \( q = 1 \) whenever \( 3p > p + 4(1 - p) \Leftrightarrow 6p > 4 \Leftrightarrow p > \frac{2}{3} \).

Row chooses \( p = 1 \) whenever \( 4q + (1 - q) > 3(1 - q) \Leftrightarrow 6q > 2 \Leftrightarrow q > \frac{1}{3} \).
There is a mixed Nash equilibrium with $p = \frac{2}{3}$ and $q = \frac{1}{3}$. 
Battle of the Sexes: Expected payoff

<table>
<thead>
<tr>
<th></th>
<th>Cafe(1/3)</th>
<th>Pub(2/3)</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe(2/3)</td>
<td>4, 3</td>
<td>1, 1</td>
<td>4 \cdot 1/3 + 2/3</td>
</tr>
<tr>
<td>Pub(1/3)</td>
<td>0, 0</td>
<td>3, 4</td>
<td>3 \cdot 2/3</td>
</tr>
<tr>
<td>Expected</td>
<td>3 \cdot 2/3</td>
<td>2/3 + 4 \cdot 1/3</td>
<td></td>
</tr>
</tbody>
</table>

Frequency of play:

<table>
<thead>
<tr>
<th></th>
<th>Cafe(1/3)</th>
<th>Pub(2/3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe(2/3)</td>
<td>2/9</td>
<td>4/9</td>
</tr>
<tr>
<td>Pub(1/3)</td>
<td>1/9</td>
<td>2/9</td>
</tr>
</tbody>
</table>

Expected utility to row player: 2
Expected utility to column player: 2
Example

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & 0, 0 & 3, 5 \\
B & 2, 2 & 3, 0 \\
\end{array}
\]

There are two pure-strategy Nash equilibria, at \((B, L)\) and \((T, R)\).

If row player places probability \(p\) on \(T\) and probability \(1 - p\) on \(B\).

\[\Rightarrow\] Column player’s best reply is to play \(L\) if \(2(1 - p) \geq 5p\), i.e., \(p \leq \frac{2}{7}\).

If column player places probability \(q\) on \(L\) and \((1 - q)\) on \(R\).

\[\Rightarrow\] \(B\) is a best reply. \(T\) is only a best reply to \(q = 0\).
There is a continuum of mixed equilibria at $\frac{2}{7} \leq p \leq 1$, all with $q = 0$. 
Example: Expected payoffs of mixed NEs

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0, 0</td>
<td>3, 5</td>
</tr>
<tr>
<td>B</td>
<td>2, 2</td>
<td>3, 0</td>
</tr>
</tbody>
</table>

Frequency of play:

<table>
<thead>
<tr>
<th>Cafe (p &gt; \frac{2}{7})</th>
<th>Cafe(0)</th>
<th>Pub(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>(p)</td>
</tr>
<tr>
<td>Pub(1 − (p))</td>
<td>0</td>
<td>1 − (p)</td>
</tr>
</tbody>
</table>

Expected utility to row player: 3

Expected utility to column player: \(5 \cdot p \in (\frac{10}{7} \approx 1.4, 5]\)
Weakly and strictly dominated strategies

<table>
<thead>
<tr>
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<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0,0</td>
<td>3,5</td>
</tr>
<tr>
<td>B</td>
<td>2,2</td>
<td>3,0</td>
</tr>
</tbody>
</table>

Note that T is weakly dominated by B.

- A weakly dominated pure strategy may play a part in a mixed (or pure) Nash equilibrium.
- A strictly dominated pure strategy cannot play a part in a Nash equilibrium!
  - Any mixed strategy which places positive weight on a strictly dominated pure strategy is itself strictly dominated. This can be seen by moving weight away from the dominated strategy.
Odd number of Nash equilibria

**Theorem (Wilson, 1970)**

Generically, any finite normal form game has an odd number of Nash equilibria.

“Generically” = if you slightly change payoffs the set of Nash equilibria does not change.
Returning to our example

<table>
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</thead>
<tbody>
<tr>
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<td>3, 5</td>
</tr>
<tr>
<td>B</td>
<td>2, 2</td>
<td>3, 0</td>
</tr>
</tbody>
</table>

There are two pure-strategy Nash equilibria, at \((B, L)\) and \((T, R)\).
There is a *continuum* of mixed equilibria at \(\frac{2}{7} \leq p \leq 1\), all with \(q = 0\).
The best-reply graph

There is a *continuum* of mixed equilibria at $\frac{2}{7} \leq p \leq 1$, all with $q = 0$. 
Example: Expected utility of mixed NEs

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & 0, 0 & 3.1, 5 \\
B & 2, 2 & 3, 0 \\
\end{array}
\]

There are two pure-strategy Nash equilibria, at \((B, L)\) and \((T, R)\).

If row player places probability \(p\) on \(T\) and probability \(1 - p\) on \(B\).

\[\Rightarrow\] Column player’s best reply is to play \(L\) if \(2(1 - p) \geq 5p\), i.e., \(p \leq \frac{2}{7}\).

If column player places probability \(q\) on \(L\) and \((1 - q)\) on \(R\).

\[\Rightarrow\] Row player’s best reply is to play \(T\) if \(3.1(1 - q) \geq 2q + 3(1 - q)\), i.e., \(q \leq 0.1/2.1\).

The unique mixed strategy equilibrium is where \(p = 2/7\) and \(q = 0.1/2.1\).
The best-reply graph

There is an odd number of equilibria.
Coordination game

<table>
<thead>
<tr>
<th></th>
<th>Email</th>
<th>Fax</th>
</tr>
</thead>
<tbody>
<tr>
<td>Email</td>
<td>5, 5</td>
<td>1, 1</td>
</tr>
<tr>
<td>Fax</td>
<td>0, 0</td>
<td>3, 4</td>
</tr>
</tbody>
</table>

The two pure Nash equilibria are \{Email, Email\} and \{Fax, Fax\}.

The unique mixed equilibrium is given by row player playing \( \sigma_1 = (1/2, 1/2) \) and column player playing \( \sigma_2 = (2/7, 5/7) \)
Invariance of Nash equilibria

**Proposition**

Any two games \( G, G' \) which differ only by a positive affine transformation of each player’s payoff function have the same set of Nash equilibria.

Adding a constant \( c \) to all payoffs of some player \( i \) which are associated with any fixed pure combination \( s_i \) for the other players sustains the set of Nash equilibria.
Coordination game

Now apply the transformation \( u' = 2 + 3 \cdot u \) to the row player’s payoffs:

\[
\begin{array}{c|cc|c|cc}
& Email & Fax & & Email & Fax \\
Email & 5, 5 & 1, 1 & & 17, 5 & 5, 1 \\
Fax & 0, 0 & 3, 4 & & 2, 0 & 11, 4 \\
\end{array}
\]

The two pure Nash equilibria remain \{Email, Email\} and \{Fax, Fax\}.

The unique mixed equilibrium is again given by row player playing \( \sigma_1 = (1/2, 1/2) \) and column player playing \( \sigma_2 = (2/7, 5/7) \).
Some remarks on Nash equilibrium

Nash equilibrium is a very powerful concept since it exists (in finite settings)!

But there are often a multitude of equilibria. Therefore game theorists ask which equilibria are more or less likely to be observed.

We will focus next on a static refinements, strict and perfect equilibrium.

Later we will talk about dynamic refinements.
Strict Nash equilibria

Definition: Strict Nash Equilibrium

A strict Nash equilibrium is a profile $\sigma^*$ such that,

$$U_i(\sigma_i^*, \sigma_{-i}^*) > U_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i$ and $i$. 
Perfect equilibrium or “trembling hand” perfection

Selten: ‘Select these equilibria which are robust to small “trembles” in the player’s strategy choices’

**Definition: \( \varepsilon \)-perfection**

Given any \( \varepsilon \in (0, 1) \), a strategy profile \( \sigma \) is \( \varepsilon \)-perfect if it is interior \((x_{ih} > 0 \text{ for all } i \in N \text{ and } h \in S_i)\) and such that:

\[
h \notin \beta_i(x) \Rightarrow x_{ih} \leq \varepsilon
\]

**Definition: Perfect equilibrium**

A strategy profile \( \sigma \) is perfect if it is the limit of some sequence of \( \varepsilon_t \)-perfect strategy profiles \( x^t \) with \( \varepsilon_t \to 0 \).
Perfect equilibrium or “trembling hand” perfection

Example:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1,1</td>
<td>1,0</td>
</tr>
<tr>
<td>B</td>
<td>1,0</td>
<td>0,0</td>
</tr>
</tbody>
</table>

There are two pure Nash equilibria $B, L$ and $T, L$. The mixed equilibrium is such that column player plays $L$ and row player plays any interior mix.

Only $T, L$ is perfect.

Note that $T, L$ is not strict.
Perfect equilibrium or “trembling hand” perfection

**Proposition (Selten 1975)**

For every finite game there exists at least one perfect equilibrium. The set of perfect equilibria is a subset of the set of Nash equilibria.

**Proposition**

Every strict equilibrium is perfect.
Dynamic games

Many situations (games) are characterized by sequential decisions and information about prior moves

- Market entrant vs. incumbent (think BlackBerry vs. Apple iPhone)
- Chess
- ...

When such a game is written in strategic form, important information about timing and information is lost.

Solution:

- Extensive form games (via game trees)
- Discussion of timing and information
- New equilibrium concepts
Example: perfect information

Battle of the sexes:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>B</td>
<td>0, 0</td>
<td>1, 3</td>
</tr>
</tbody>
</table>

What if row player (player 1) can decide first?
Example: perfect information

What would you do as player 1, A or B?
What would you do as player 2 if player 1 played A, a or b?
What would you do as player 2 if player 1 played B, a or b?
Example: perfect information

Player 2 would like to commit that if player 1 plays A he will play b (in order to make player 1 play B).
But fighting is not time consistent. Once player 1 played A it is not rational for player 2 to play b.
The expected outcome is A followed by a for payoffs (3, 1).
This is called **backward induction**. It results in a **subgame perfect equilibrium**. More later!
Example: imperfect information

What would you do as player 1, A or B?
What would you do as player 2, a or b?

Timing and information matters!
Extensive form game: Definition

An **extensive-form game** is defined by:

- **Players**, \( N = \{1, \ldots, n\} \), with typical player \( i \in N \). Note: *Nature* can be one of the players.

- Basic structure is a tree, the **game tree** with nodes \( a \in A \). Let \( a_0 \) be the root of the tree.

- Nodes are game states which are either
  - **Decision nodes** where some player chooses an action
  - **Chance nodes** where nature plays according to some probability distribution
Representation

Extensive form
- Directed graph with single initial node; edges represent moves
- Probabilities on edges represent Nature moves
- Nodes that the player in question cannot distinguish (information sets) are circled together (or connected by dashed line)

Extensive form $\rightarrow$ normal form
- A strategy is a player’s complete plan of action, listing move at every information set of the player
- Different extensive form games may have same normal form (loss of information on timing and information)

Question: What is the number of a player’s strategies?
Product of the number of actions available at each of his information sets.
Subgames (Selten 1965, 1975)

Given a node $a$ in the game tree consider the subtree rooted at $a$. $a$ is the root of a subgame if

- $a$ is the only node in its information set
- if a node is contained in the subgame then all its successors are contained in the subgame
- every information set in the game either consists entirely of successor nodes to $a$ or contains no successor node to $a$.

If a node $a$ is a subroot, then each player, when making a choice at any information set in the game, knows whether $a$ has been reached or not. Hence if $a$ has been reached it is as if a “new” game has started.
Subgame examples

How many subgames does the game have?

Which strategies does each player have?

Strategies player 1: \{A, B\}

Strategies player 2: \{(a, a), (a, b), (b, a), (b, b)\}
Subgame examples

How many subgames does the game have?

Which strategies does each player have?

Strategies player 1: \{A, B\}
Strategies player 2: \{a, b\}
Subgame examples: Equivalence to normal form

where columns strategies are of the form *strategy against A, strategy against B*
Strategies in extensive games

**Pure strategy** $s_i$  One move for each information set of the player.

**Mixed strategy** $\sigma_i$  Any probability distribution $x_i$ over the set of pure strategies $S_i$.

**Behavior strategy** $y_i$  Select randomly at each information set the move to be made (can delay coin-toss until getting there).

Behavior strategies are special case of a mixed strategy: moves are made with independent probabilities at information sets.

Pure strategies are special case of a behavior strategy.
Example (imperfect recall)

There is one player who has “forgotten” his first move when his second move comes up. (For example: did he lock the door before leaving or not?)

The indicated outcome, with probabilities in brackets, results from the mixed strategy, $\frac{1}{2}Aa + \frac{1}{2}Bb$.

$\Rightarrow$ There exists no behavior strategy that induces this outcome.

The player exhibits “poor memory” / “imperfect recall”.

Perfect recall

Perfect recall (Kuhn 1950)

Player $i$ in an extensive form game has *perfect recall* if for every information set $h$ of player $i$, all nodes in $h$ are preceded by the same sequence of moves of player $i$. 
Kuhn’s theorem

**Definition: Realization equivalent**

A mixed strategy $\sigma_i$ is *realization equivalent* with a behavior strategy $y_i$ if the realization probabilities under the profile $\sigma_i, \sigma_{-i}$ are the same as those under $y_i, \sigma_{-i}$ for all profiles $\sigma$.

**Kuhn’s theorem**

Consider a player $i$ in an extensive form with perfect recall. For every mixed strategy $\sigma_i$ there exists a realization-equivalent behavior strategy $y_i$. 
Kuhn’s Theorem - proof (not part of exam)

Given: mixed strategy \( \sigma \)
Wanted: realization equivalent behavior strategy \( y \)

Idea: \( y = \text{observed behavior under } \sigma \)
\( y(c) = \text{observed probability } \sigma(c) \) of making move \( c \).

What is \( \sigma(c) \)?

Look at sequence ending in \( c \), here \( lbc \).
\( \sigma[lbc] = \text{probability of } lbc \text{ under } \sigma = \sigma(l, b, c). \)

Sequence \( lb \) leading to info set \( h \)

\[
\mu[lb] = \sigma(l, b, c) + \sigma(l, b, d)
\]
\[\Rightarrow \sigma[lb] = \sigma[lbc] + \sigma[lbd]\]
Kuhn’s Theorem - proof (not part of exam)

\[ \Rightarrow \sigma(c) = \frac{\sigma[lbc]}{\sigma[lb]} =: y(c) \]

\[ \Rightarrow \sigma(b) = \frac{\sigma[l]}{\sigma[l]} =: y(b) \]

first info set: \( \sigma[\emptyset] = 1 = \sigma[l] + \sigma[r] \)

\[ \sigma(l) = \frac{\sigma[l]}{\sigma[\emptyset]} =: y(l) \]

\[ \Rightarrow y(l)y(b)y(c) = \frac{\sigma[l]}{\sigma[\emptyset]} \cdot \frac{\sigma[l]}{\sigma[l]} \cdot \frac{\sigma[lbc]}{\sigma[lb]} \]

\[ = \sigma[lbc] \]

\[ \Rightarrow y \text{ equivalent to } \sigma \]
Subgame perfect equilibrium

**Definition: subgame perfect equilibrium (Selten 1965)**

A behavior strategy profile in an extensive form game is a *subgame perfect equilibrium* if for each subgame the restricted strategy is a Nash equilibrium of the subgame.

**Theorem**

Every finite game with perfect recall has at least one subgame perfect equilibrium. Generic such games have a unique subgame perfect equilibrium.

Generic = with probability 1 when payoffs are drawn from continuous independent distributions.
Example: An Outside-option game

Reconsider the battle-of-sexes game (BS game), but player 1 can decide if she joins the game before.

What are the subgames?
What are the subgame perfect equilibria?
Example: An Outside-option game

If player 1 decides to enter the BS subgame, player 2 will know that player 1 joint, but will not know her next move. There exist three subgame perfect equilibria, one for each Nash equilibria of the BS game:

- \( S = \{EA, A\} \) Player 1 earns 3, Player 2 earns 1.
- \( S = \{TB, B\} \) Player 1 earns 2, Player 2 earns \(-1\).
- \( S = \{T(3/4 \cdot A + 1/4 \cdot B), (1/4 \cdot A + 3/4 \cdot B)\} \) Player 1 earns 2, Player 2 earns \(-1\).
“Reasoning backwards in time”:

- First consider the last time a decision might be made and choose what to do (that is, find Nash equilibria) at that time.
- Using the former information, consider what to do at the second-to-last time a decision might be made.
- ...
- This process terminates at the beginning of the game, the found behavior strategies are subgame perfect equilibria.
Example: The Centipede game (Rosenthal)

What is the unique subgame perfect equilibrium?

Stop at all nodes.

But in experiments most subjects Pass initially: a "trust bubble" forms.

Palacios-Huerta & Volij:

- Chess masters stop right away; students do not...
- ...unless they are told they are playing chess masters.
THANKS EVERYBODY
See you next week!
And keep checking the website for new materials as we progress:
http://www.coss.ethz.ch/education/GT.html