NORMAL FORM GAMES: invariance and refinements
DYNAMIC GAMES: extensive form

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Plan

- **Normal form games**
  - Equilibrium invariance
  - Equilibrium refinements

- **Dynamic games**
  - Extensive form games
  - Incomplete information
  - Sub-game perfection
Nash’s equilibrium existence theorem

**Theorem (Nash 1951)**

Every finite game has at least one [Nash] equilibrium in mixed strategies.
Cook book: How to find mixed Nash equilibria

- Find all pure strategy NE.

Check whether there is an equilibrium in which row mixes between several of her strategies:

- Identify candidates:
  - If there is such an equilibrium then each of these strategies must yield the same expected payoff given column’s equilibrium strategy.
  - Write down these payoffs and solve for column’s equilibrium mix.
  - Reverse: Look at the strategies that column is mixing on and solve for row’s equilibrium mix.

- Check candidates:
  - The equilibrium mix we found must indeed involve the strategies for row we started with.
  - All probabilities we found must indeed be probabilities (between 0 and 1).
  - Neither player has a positive deviation.
Battle of the Sexes revisited

**Players**  The players are the two students $N = \{\text{row, column}\}$.

**Strategies**  Row chooses from $S_{\text{row}} = \{\text{Cafe, Pub}\}$
Column chooses from $S_{\text{column}} = \{\text{Cafe, Pub}\}$.

**Payoffs**  For example, $u_{\text{row}}(\text{Cafe, Cafe}) = 4$. The following matrix summarises:

<table>
<thead>
<tr>
<th></th>
<th>Cafe($q$)</th>
<th>Pub($1-q$)</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe($p$)</td>
<td>4, 3</td>
<td>1, 1</td>
<td>$4q + (1-q)$</td>
</tr>
<tr>
<td>Pub($1-p$)</td>
<td>0, 0</td>
<td>3, 4</td>
<td>$3(1-q)$</td>
</tr>
<tr>
<td>Expected</td>
<td>3$p$</td>
<td>$p + 4(1-p)$</td>
<td></td>
</tr>
</tbody>
</table>

Column chooses $q = 1$ whenever $3p > p + 4(1 - p) \Leftrightarrow 6p > 4 \Leftrightarrow p > \frac{2}{3}$.

Row chooses $p = 1$ whenever $4q + (1 - q) > 3(1 - q) \Leftrightarrow 6q > 2 \Leftrightarrow q > \frac{1}{3}$. 
Battle of the Sexes: Best-reply graph

There is a mixed Nash equilibrium with \( p = \frac{2}{3} \) and \( q = \frac{1}{3} \).
Battle of the Sexes: Expected payoff

<table>
<thead>
<tr>
<th></th>
<th>Cafe(1/3)</th>
<th>Pub(2/3)</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe(2/3)</td>
<td>4, 3</td>
<td>1, 1</td>
<td>4·1/3 + 2/3</td>
</tr>
<tr>
<td>Pub(1/3)</td>
<td>0, 0</td>
<td>3, 4</td>
<td>3·2/3</td>
</tr>
<tr>
<td>Expected</td>
<td>3 · 2/3</td>
<td>2/3 + 4 · 1/3</td>
<td></td>
</tr>
</tbody>
</table>

Frequency of play:

<table>
<thead>
<tr>
<th></th>
<th>Cafe(1/3)</th>
<th>Pub(2/3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cafe(2/3)</td>
<td>2/9</td>
<td>4/9</td>
</tr>
<tr>
<td>Pub(1/3)</td>
<td>1/9</td>
<td>2/9</td>
</tr>
</tbody>
</table>

Expected utility to row player: 2
Expected utility to column player: 2
Example

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>0, 0</td>
<td>3, 5</td>
</tr>
<tr>
<td>$B$</td>
<td>2, 2</td>
<td>3, 0</td>
</tr>
</tbody>
</table>
The best-reply graph
Example: Expected payoffs of mixed NEs
Weakly and strictly dominated strategies

\[
\begin{array}{c|cc}
  & L & R \\
\hline
T & 0, 0 & 3, 5 \\
B & 2, 2 & 3, 0 \\
\end{array}
\]

Note that \( T \) is weakly dominated by \( B \).

- A weakly dominated pure strategy may play a part in a mixed (or pure) Nash equilibrium.
- A strictly dominated pure strategy cannot play a part in a Nash equilibrium!
  - Any mixed strategy which places positive weight on a strictly dominated pure strategy is itself strictly dominated. This can be seen by moving weight away from the dominated strategy.
Odd number of Nash equilibria

**Theorem (Wilson, 1970)**

Generically, any finite normal form game has an odd number of Nash equilibria.

“Generically” = if you slightly change payoffs the set of Nash equilibria does not change.
Returning to our example

<table>
<thead>
<tr>
<th></th>
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<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0, 0</td>
<td>3, 5</td>
</tr>
<tr>
<td>B</td>
<td>2, 2</td>
<td>3, 0</td>
</tr>
</tbody>
</table>

There are two pure-strategy Nash equilibria, at \((B, L)\) and \((T, R)\).

There is a continuum of mixed equilibria at \(\frac{2}{7} \leq p \leq 1\), all with \(q = 0\).
The best-reply graph

There is a \textit{continuum} of mixed equilibria at $\frac{2}{7} \leq p \leq 1$, all with $q = 0$. 
Example: Expected utility of mixed NEs

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0, 0</td>
<td>3, 1, 5</td>
</tr>
<tr>
<td>B</td>
<td>2, 2</td>
<td>3, 0</td>
</tr>
</tbody>
</table>

There are two pure-strategy Nash equilibria, at \((B, L)\) and \((T, R)\).

If row player places probability \(p\) on \(T\) and probability \(1 - p\) on \(B\).
⇒ Column player’s best reply is to play \(L\) if \(2(1 - p) \geq 5p\), i.e., \(p \leq \frac{2}{7}\).

If column player places probability \(q\) on \(L\) and \((1 - q)\) on \(R\).
⇒ Row player’s best reply is to play \(T\) if \(3.1(1 - q) \geq 2q + 3(1 - q)\), i.e., \(q \leq 0.1/2.1\).

The unique mixed strategy equilibrium is where \(p = 2/7\) and \(q = 0.1/2.1\).
The best-reply graph

There is an odd number of equilibria.
Coordinating game

<table>
<thead>
<tr>
<th></th>
<th>Email</th>
<th>Fax</th>
</tr>
</thead>
<tbody>
<tr>
<td>Email</td>
<td>5,5</td>
<td>1,1</td>
</tr>
<tr>
<td>Fax</td>
<td>0,0</td>
<td>3,4</td>
</tr>
</tbody>
</table>
Invariance of Nash equilibria

Proposition

Any two games $G, G'$ which differ only by a positive affine transformation of each player’s payoff function have the same set of Nash equilibria.

Adding a constant $c$ to all payoffs of some player $i$ which are associated with any fixed pure combination $s_i$ for the other players sustains the set of Nash equilibria.
Coordination game

Now apply the transformation $u' = 2 + 3 \cdot u$ to the row player’s payoffs:

<table>
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<th>Email</th>
<th>Fax</th>
</tr>
</thead>
<tbody>
<tr>
<td>Email</td>
<td>5, 5</td>
<td>1, 1</td>
</tr>
<tr>
<td>Fax</td>
<td>0, 0</td>
<td>3, 4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Email</th>
<th>Fax</th>
</tr>
</thead>
<tbody>
<tr>
<td>Email</td>
<td>17, 5</td>
<td>5, 1</td>
</tr>
<tr>
<td>Fax</td>
<td>2, 0</td>
<td>11, 4</td>
</tr>
</tbody>
</table>
Some remarks on Nash equilibrium

Nash equilibrium is a very powerful concept since it exists (in finite settings)!

But there are often a multitude of equilibria. Therefore game theorists ask which equilibria are more or less likely to be observed.

We will focus next on a static refinements, strict and perfect equilibrium.

Later we will talk about dynamic refinements.
Strict Nash equilibria

**Definition: Strict Nash Equilibrium**

A *strict Nash equilibrium* is a profile $\sigma^*$ such that,

$$U_i(\sigma^*_i, \sigma^*_{-i}) > U_i(\sigma_i, \sigma^*_{-i})$$

for all $\sigma_i$ and $i$. 
Perfect equilibrium or “trembling hand” perfection

Selten: ‘Select these equilibria which are robust to small “trembles” in the player’s strategy choices’

**Definition: ε-perfection**

Given any \( \varepsilon \in (0, 1) \), a strategy profile \( \sigma \) is \( \varepsilon \)-perfect if it is interior \( (x_{ih} > 0 \text{ for all } i \in N \text{ and } h \in S_i) \) and such that:

\[
h \notin \beta_i(x) \Rightarrow x_{ih} \leq \varepsilon
\]

**Definition: Perfect equilibrium**

A strategy profile \( \sigma \) is perfect if it is the limit of some sequence of \( \varepsilon_t \)-perfect strategy profiles \( x^t \) with \( \varepsilon_t \rightarrow 0 \).
Perfect equilibrium or “trembling hand” perfection

Example:

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & 1, 1 & 1, 0 \\
B & 1, 0 & 0, 0 \\
\end{array}
\]

There are two pure Nash equilibria \( B, L \) and \( T, L \). The mixed equilibrium is such that column player plays \( L \) and row player plays any interior mix.

Only \( T, L \) is perfect.

Note that \( T, L \) is not strict.
Perfect equilibrium or “trembling hand” perfection

Proposition (Selten 1975)
For every finite game there exists at least one perfect equilibrium. The set of perfect equilibria is a subset of the set of Nash equilibria.

Proposition
Every strict equilibrium is perfect.
Dynamic games

Many situations (games) are characterized by sequential decisions and information about prior moves

- Market entrant vs. incumbent (think BlackBerry vs. Apple iPhone)
- Chess
- ...

When such a game is written in strategic form, important information about timing and information is lost.

Solution:

- Extensive form games (via game trees)
- Discussion of timing and information
- New equilibrium concepts
Example: perfect information

Battle of the sexes:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3,1</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,3</td>
</tr>
</tbody>
</table>

What if row player (player 1) can decide first?
Example: perfect information

What would you do as player 1, A or B?
What would you do as player 2 if player 1 played A, a or b?
What would you do as player 2 if player 1 played B, a or b?
Example: perfect information
Example: imperfect information

What would you do as player 1, A or B?
What would you do as player 2, a or b?

Timing and information matters!
Extensive form game: Definition

An **extensive-form game** is defined by:

- **Players**, \( N = \{1, \ldots, n\} \), with typical player \( i \in N \). Note: *Nature* can be one of the players.

- Basic structure is a tree, the **game tree** with nodes \( a \in A \). Let \( a_0 \) be the root of the tree.

- Nodes are game states which are either
  - **Decision nodes** where some player chooses an action
  - **Chance nodes** where nature plays according to some probability distribution
Representation

Extensive form

- Directed graph with single initial node; edges represent moves
- Probabilities on edges represent Nature moves
- Nodes that the player in question cannot distinguish (information sets) are circled together (or connected by dashed line)

Extensive form $\rightarrow$ normal form

- A strategy is a player’s complete plan of action, listing move at every information set of the player
- Different extensive form games may have same normal form (loss of information on timing and information)

Question: What is the number of a player’s strategies?
Product of the number of actions available at each of his information sets.
Subgames (Selten 1965, 1975)

Given a node $a$ in the game tree consider the subtree rooted at $a$. $a$ is the root of a subgame if

- $a$ is the only node in its information set
- if a node is contained in the subgame then all its successors are contained in the subgame
- every information set in the game either consists entirely of successor nodes to $a$ or contains no successor node to $a$.

If a node $a$ is a subroot, then each player, when making a choice at any information set in the game, knows whether $a$ has been reached or not. Hence if $a$ has been reached it is as if a “new” game has started.
Subgame examples

How many subgames does the game have?

\[
\begin{array}{cccc}
(3,1) & (0,0) & (0,0) & (1,3) \\
a & b & a & b \\
2 & 2 & 2 & \\
A & B & \\
1 & & & \\
\end{array}
\]
Subgame examples

How many subgames does the game have?
**Subgame examples: Equivalence to normal form**

where columns strategies are of the form *strategy against A, strategy against B*
Strategies in extensive games

**Pure strategy** $s_i$ One move for each information set of the player.

**Mixed strategy** $\sigma_i$ Any probability distribution $x_i$ over the set of pure strategies $S_i$.

**Behavior strategy** $y_i$ Select randomly at each information set the move to be made (can delay coin-toss until getting there).

Behavior strategies are special case of a mixed strategy: moves are made with independent probabilities at information sets.

Pure strategies are special case of a behavior strategy.
Example (imperfect recall)

There is one player who has “forgotten” his first move when his second move comes up. (For example: did he lock the door before leaving or not?)

The indicated outcome, with probabilities in brackets, results from the mixed strategy, \( \frac{1}{2}Aa + \frac{1}{2}Bb. \)

\[ \Rightarrow \] There exists no behavior strategy that induces this outcome.

The player exhibits “poor memory” / “imperfect recall”. 
Perfect recall

Perfect recall (Kuhn 1950)

Player $i$ in an extensive form game has *perfect recall* if for every information set $h$ of player $i$, all nodes in $h$ are preceded by the same sequence of moves of player $i$. 
Kuhn’s theorem

Definition: Realization equivalent

A mixed strategy $\sigma_i$ is realization equivalent with a behavior strategy $y_i$ if the realization probabilities under the profile $\sigma_i, \sigma_{-i}$ are the same as those under $y_i, \sigma_{-i}$ for all profiles $\sigma$.

Kuhn’s theorem

Consider a player $i$ in an extensive form with perfect recall. For every mixed strategy $\sigma_i$ there exists a realization-equivalent behavior strategy $y_i$. 
Kuhn’s Theorem - proof (not part of exam)

Given: mixed strategy $\sigma$
Wanted: realization equivalent behavior strategy $y$

Idea: $y = \text{observed behavior under } \sigma$
$y(c) = \text{observed probability } \sigma(c) \text{ of making move } c$.

What is $\sigma(c)$?

Look at sequence ending in $c$, here $Ibc$.
$\sigma[lbc] = \text{probability of } lbc \text{ under } \sigma = \sigma(l, b, c)$.

Sequence $lb$ leading to info set $h$

$$\mu[lb] = \sigma(l, b, c) + \sigma(l, b, d)$$
$$\Rightarrow \sigma[lb] = \sigma[lbc] + \sigma[lbd]$$
Kuhn’s Theorem - proof (not part of exam)

\[ \Rightarrow \sigma(c) = \frac{\sigma[lbc]}{\sigma[lb]} =: y(c) \]

\[ \Rightarrow \sigma(b) = \frac{\sigma[l]}{\sigma[l]} =: y(b) \]

first info set: \[ \sigma[\emptyset] = 1 = \sigma[l] + \sigma[r] \]

\[ \sigma(l) = \frac{\sigma[l]}{\sigma[\emptyset]} =: y(l) \]

\[ \Rightarrow y(l)y(b)y(c) = \frac{\sigma[l]}{\sigma[\emptyset]} \cdot \frac{\sigma[l]}{\sigma[l]} \cdot \frac{\sigma[lbc]}{\sigma[lb]} \]

\[ = \sigma[lbc] \]

\[ \Rightarrow y \text{ equivalent to } \sigma \]
Subgame perfect equilibrium

Definition: subgame perfect equilibrium (Selten 1965)

A behavior strategy profile in an extensive form game is a subgame perfect equilibrium if for each subgame the restricted strategy is a Nash equilibrium of the subgame.

Theorem

Every finite game with perfect recall has at least one subgame perfect equilibrium. Generic such games have a unique subgame perfect equilibrium.

Generic = with probability 1 when payoffs are drawn from continuous independent distributions.
Example: An Outside-option game

Reconsider the battle-of-sexes game (BS game), but player 1 can decide if she joins the game before.

What are the subgames?

What are the subgame perfect equilibria?
Example: An Outside-option game
Cook-book: Backward induction

“Reasoning backwards in time”:

- First consider the last time a decision might be made and choose what to do (that is, find Nash equilibria) at that time
- Using the former information, consider what to do at the second-to-last time a decision might be made
- ...
- This process terminates at the beginning of the game, the found behavior strategies are subgame perfect equilibria
Example: The Centipede game (Rosenthal)

What is the unique subgame perfect equilibrium?

Stop at all nodes.

But in experiments most subjects Pass initially: a "trust bubble" forms.

Palacios-Huerta & Volij:

- Chess masters stop right away; students do not...
- ...unless they are told they are playing chess masters.
THANKS EVERYBODY
See you next week!
And keep checking the website for new materials as we progress:
http://www.coss.ethz.ch/education/GT.html